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Lusztig isomorphisms for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type

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ABSTRACT

In the structure theory of quantized enveloping algebras, the algebra isomorphisms determined by Lusztig led to the first general construction of PBW bases of these algebras. Also, they have important applications to the representation theory of these and related algebras. In the present paper the Drinfel'd double for a class of graded Hopf algebras is investigated. Various quantum algebras, including small quantum groups and multiparameter quantizations of semisimple Lie algebras and of Lie superalgebras, are covered by the given definition. For these Drinfel'd doubles Lusztig maps are defined. It is shown that these maps induce isomorphisms between doubles of bosonizations of Nichols algebras of diagonal type. Further, the obtained isomorphisms satisfy Coxeter type relations in a generalized sense. As an application, the Lusztig isomorphisms are used to give a characterization of Nichols algebras of diagonal type with finite root system.

1. Historical remarks

The emergence of quantum groups following the work of Drinfel'd [Dri87] and Jimbo [Jim86] was characterized by the appearance of a huge amount of papers considering generalizations of quantized enveloping algebras of semisimple Lie algebras, their structure theory, and their applications in physics and mathematics. One of the remarkable discoveries with far reaching consequences in the field was Lusztig's construction of automorphisms of $U_q(\mathfrak{g})$, see [Lus93]. It led to the construction of Poincaré–Birkhoff–Witt (PBW) bases of $U_q(\mathfrak{g})$ and to the study of crystal bases. Lusztig's isomorphisms are also very important for the representation theory of quantized enveloping algebras.

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As a particular type of generalization of quantized enveloping algebras, in the early 1990s quantized enveloping algebras of contragredient Lie superalgebras have been intensively studied, see e.g. [KT91,FLV91,KT95,BKM98,Yam99]. First, as noted in the introduction of [KT91], it was not clear whether there is an appropriate structure which could play a similar role for quantized Lie superalgebras as the Weyl group does for quantized semisimple Lie algebras. After the appearance of Serganova's work [Ser96] on generalized root systems the idea of a Weyl groupoid and corresponding Lusztig isomorphisms were mentioned by Khoroshkin and Tolstoy [KT95, p. 16] and used implicitly by Yamane [Yam99, Sections 7.5,8], [Yam01] in a topological setting. Presumably because of technical difficulties the response on these papers was not very high, and a more detailed elaboration of these structures is still missing. As a result of the project aiming the classification of finite-dimensional Nichols algebras of diagonal type, the Weyl groupoid was rediscovered in a more general context [Hec06], based on a natural definition of generalized root systems (different from the one of Serganova). In the meantime, a complete list of Nichols algebras of diagonal type with finite root system [Hec09] was determined and a piece of an appealing structure theory of generalized root systems and Weyl groupoids [HY08,CH09] is available. A very interesting perspective for the future is the existence of root systems and Weyl groupoids for a much larger class of Hopf algebras [HS08].

Recently, for two-parameter quantizations of finite-dimensional simple Lie algebras, Bergeron, Gao, and Hu [BGH06] study generalizations of Lusztig isomorphisms. In the rank two case they are able to define such isomorphisms in the full generality of their approach [BGH06, Section 3]. In the present paper it is shown how to use the Weyl groupoid for the definition of Lusztig isomorphisms for a large class of quantum doubles, including (standard and) multiparameter quantizations of enveloping algebras of semisimple Lie algebras and Lie superalgebras and their small quantum group analogs. An important fact is that the use of the Weyl groupoid removes most of the technical assumptions in the definition of the quantum doubles under investigation.

In the case of Lie superalgebras and their quantized analogs a new phenomenon compared to semisimple Lie algebras arises. Namely, (quantum) Serre relations are not sufficient to define the Lie (or quantized enveloping) superalgebra by generators and relations, see [FLV91] and [KT91]. The determination of a minimal set of defining relations turned out to be solvable in principal by using the Weyl groupoid — see [Yam99] for explicit lists and [GL01, Section 3.5] for some ideas — but it involves technical difficulties. In the classical case computations were done by Grozman and Leites [GL01], and Yamane [Yam99]. The latter paper also treats the quantum case for its topological version. The fact that the papers [KT91,GL01,Yam99] give different sets of defining relations, shows that a description avoiding case by case considerations would be of advantage for further study of the subject.

The Weyl groupoid turned out to be the key structure to answer the first part of [And02, Question 5.9], namely, to determine all finite-dimensional Nichols algebras of diagonal type. In view of the results discussed above it seems that the second part of [And02, Question 5.9], which asks for the defining relations of these algebras, can be answered in its naive sense — by giving explicit lists — only in a very technical way. A possible application of Lusztig isomorphisms and their properties is to give an answer to [And02, Question 5.9] in a conceptual way based on the idea described in [GL01, Section 3.5] for contragredient Lie superalgebras.

2. On the structure of this paper

The mathematical part of the paper starts in the next section with recalling some combinatorial aspects of Nichols algebras of diagonal type. The Weyl groupoid and the root system of a bicharacter are at the heart of the structure theory of finite-dimensional (and also more general) Nichols algebras of diagonal type, and they will appear on many places in the paper. Then in Section 4 the Drinfel'd double $\mathcal{U}(\chi)$ of the tensor algebra $\mathcal{U}^+(\chi)$ of a braided vector space of diagonal type, see Definition 4.5 and Proposition 4.6, is studied. For the convenience of the reader, many facts known from the theory of quantized enveloping algebras and superalgebras are worked out explicitly in the presented more general context. The style of the presentation and the notation follow the conventions in standard textbooks on quantum groups. In this section, more precisely in Proposition 4.17, a characterization of ideals of $\mathcal{U}(\chi)$ admitting a triangular decomposition of the corresponding quotient algebra is proven, which seems to be new even for multiparameter quantizations of Kac–Moody algebras, see [KS08, Proposition 3.4].

In Section 5 the definition and structure of Nichols algebras is recalled. Most facts appear in some form in the literature.

The main part of the paper starts in Section 6. There are two important aims chased from now on. First, for a class of Drinfel'd doubles $U(\chi)$ of Nichols algebras of diagonal type the definition of Lusztig isomorphisms is given in Theorem 6.11. For this definition a combinatorial restriction on χ is indispensable, as explained at the beginning of Section 6.1. The idea behind this condition is that the Lusztig isomorphisms are natural realizations of elements of the Weyl groupoid $\mathcal{W}(\chi)$ attached to the bicharacter χ, see Definition 3.12, and the definition of the generating reflections of the Weyl groupoid requires a finiteness condition on χ. The proof of the fact, that the Lusztig maps are indeed well defined and isomorphisms, requires several intermediate results. Therefore, and in order to obtain statements in a more general setting, the Lusztig maps T_p and T_p^- are introduced in the most universal setting in Lemma 6.6. Besides the obvious analogy to Lusztig's definition, the main difference is the missing of the constant factors in $T_p(E_i)$. The advantage of this modification is that one can avoid case by case checkings in the proofs of *all* of the results concerning the maps T_p in this paper. This is not a negligible fact in view of [Lus93, Subsection 39.2] and the classification result for Nichols algebras with finite root system in [Hec09], even if one restricts himself to the rank 2 cases. However, the given definition has also its disadvantage: In equations as for example Eq. (6.14) and Eq. (6.25) one cannot remove the field $\Bbbk$. This implies in particular that the Coxeter relations in Theorem 6.19 hold "only" up to an automorphism $\varphi_{\underline{a}}$ defined in Proposition 4.9(1).

The main results concerning Lusztig isomorphisms are variants of the corresponding statements for quantized enveloping algebras of semisimple Lie algebras.

- Proposition 6.8: The Lusztig maps induce isomorphisms between the Drinfel'd doubles of the corresponding Nichols algebras of diagonal type.
- Theorem 6.19: Lusztig isomorphisms satisfy Coxeter type relations, up to a natural automorphism of $U(\chi)$.
- Theorem 6.20: The images of certain generators under a Lusztig isomorphism are in the upper triangular part of the Drinfel'd double.
- Corollary 6.21: Description of the Lusztig isomorphism corresponding to a longest element of the Weyl groupoid.

The other important aim of the main part of the paper is to give a characterization of Nichols algebras of diagonal type having a finite root system. The corresponding result is Theorem 7.1. The theorem claims, roughly speaking, that a "natural" ideal $\mathcal{I}^+(\chi)$ of $\mathcal{U}^+(\chi)$ is the defining ideal of the Nichols algebra $U^+(\chi)$ if and only if for all $\chi' \in \mathrm{Ob}(\mathcal{W}(\chi))$ there exist further "natural" ideals $\mathcal{I}^+(\chi')$ of $\mathcal{U}^+(\chi')$, such that all Lusztig maps between the corresponding quotient algebras are well defined. This theorem is descriptive, and admits to check whether a given family of ideals defines the corresponding family of Nichols algebras. However, it does not tell how to construct a minimal set of generators for the defining ideal of the Nichols algebra. This problem remains open for further research.

The paper ends with an application of Theorem 7.1. More precisely, in Example 7.4 it is proven that the Nichols algebra $U^+(\chi)$ associated to a bicharacter of finite Cartan type is, if the main parameter is not a root of 1, defined by Serre relations only. This result is standard in the case of usual quantized enveloping algebras.

If not indicated otherwise, all algebras in the text will be defined over a base field $\Bbbk$ of arbitrary characteristic, and they are associative and have a unit. The coproduct, counit, and antipode of a Hopf algebra will be denoted by Δ, ε, and S, respectively. For the coproduct of a Hopf algebra H the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for all $h \in H$ will be used. In contrast, for the coproduct $\underline{\Delta}$ of a braided Hopf algebra H' we follow the modified Sweedler notation of Andruskiewitsch and Schneider, see the end of the introduction in [AS02b], in form of $\underline{\Delta}(h) = h^{(1)} \otimes h^{(2)}$ for all $h \in H'$. For an arbitrary coalgebra C let C^{cop} denote the vector space C together with the coproduct opposite to the one of C.

The antipode of Hopf algebras and braided Hopf algebras is always meant to be bijective. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and positive integers, respectively, and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3. Preliminaries

Let $\Bbbk$ be a field and let $\Bbbk^\times = \Bbbk \setminus \{0\}$. For any non-empty finite set I let $\{\alpha_i \mid i \in I\}$ denote the standard basis of $\mathbb{Z}^I$.

3.1. *q*-binomial coefficients

The assertions and formulas in this subsection are analogs of those in standard textbooks on quantum groups, see for example [Lus93, Sections 1.3, 34.1], [Jos95, Sections 1.2.12–1.2.13], and [KS97, Section 2.1].

Let $q \in \Bbbk^\times$. Set $(0)_q = 0$ and for all $m \in \mathbb{N}$ let

$$(m)_q = 1 + q + \cdots + q^{m-1}, \qquad (-m)_q = -(m)_q. \tag{3.1}$$

Let $(0)_q^! = 1$ and for all $m \in \mathbb{N}$ let $(m)_q^! = \prod_{n=1}^m (n)_q$.

The quantum plane is the unital associative $\Bbbk$-algebra

$$\Bbbk\langle u, v\rangle/(vu - quv).$$

The set $\{u^m v^n \mid m, n \in \mathbb{N}_0\}$ is a $\Bbbk$-basis of this algebra. For all $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ define $\binom{m}{n}_q \in \Bbbk$ by the equation

$$(u+v)^m = \sum_{n \in \mathbb{Z}} \binom{m}{n}_q u^n v^{m-n}.$$

Since $(u+v)^{m+1} = (u+v)(u+v)^m = (u+v)^m(u+v)$, one obtains that

$$\binom{m}{n-1}_q + q^n \binom{m}{n}_q = q^{m-n+1}\binom{m}{n-1}_q + \binom{m}{n}_q = \binom{m+1}{n}_q \tag{3.2}$$

for all $m \in \mathbb{N}_0$, $n \in \mathbb{Z}$. As a special case one gets

$$\binom{m}{n}_q = 0 \quad \text{for } n < 0 \text{ or } n > m,$$

$$\binom{m}{0}_q = \binom{m}{m}_q = 1, \qquad \binom{m}{1}_q = \binom{m}{m-1}_q = (m)_q.$$

Lemma 3.1. *Let $q \in \Bbbk^\times$, $m \in \mathbb{N}_0$, and $n \in \mathbb{Z}$. Then*

$$(n+1)_q \binom{m}{n+1}_q = (m-n)_q \binom{m}{n}_q.$$

Proof. Proceed by induction on m. If $m = 0$, then both sides of the equation are zero for all $n \in \mathbb{Z}$. Suppose now that the claim holds for some $m \in \mathbb{N}_0$. Then Eq. (3.2) and the induction hypothesis imply that

$$
\begin{aligned}
(n+1)_q\binom{m+1}{n+1}_q &= (n+1)_q\left(\binom{m}{n}_q + q^{n+1}\binom{m}{n+1}_q\right)\\
&= q^n\binom{m}{n}_q + (n)_q\binom{m}{n}_q + q^{n+1}(n+1)_q\binom{m}{n+1}_q\\
&= q^n\binom{m}{n}_q + (m-n+1)_q\binom{m}{n-1}_q + q^{n+1}(m-n)_q\binom{m}{n}_q\\
&= q^n(m-n+1)_q\binom{m}{n}_q + (m-n+1)_q\binom{m}{n-1}_q\\
&= (m-n+1)_q\binom{m+1}{n}_q
\end{aligned}
$$

for all $n \in \mathbb{Z}$. This proves the lemma. $\square$

Lemma 3.2. *Let $q \in \Bbbk^\times$ and $m \in \mathbb{N}$. Assume that $(m)_q = 0$ and $(m-1)_q^! \neq 0$. Then $\binom{m}{n}_q = 0$ for all $n \in \mathbb{N}$ with $n < m$.*

Proof. The assumption yields that $\binom{m}{1}_q = (m)_q = 0$. Using Lemma 3.1 the claim follows easily by induction on n. $\square$

3.2. Cartan schemes, Weyl groupoids, and root systems

The combinatorics of the Drinfel'd double of the bosonization of a Nichols algebra of diagonal type is controlled to a large extent by its Weyl groupoid. Here the language developed in [CH09] is used. Substantial part of the theory was obtained first in [HY08]. In this subsection the most important definitions and facts are recalled.

Let I be a non-empty finite set. By [Kac90, Section 1.1] a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leqslant 0$ for all $i, j, k \in I$ with $j \neq k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 3.3. Let I be a non-empty finite set, A a non-empty set, $r_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}\big(I, A, (r_i)_{i\in I}, \big(C^a\big)_{a\in A}\big)$$

is called a *Cartan scheme* if

(C1) $r_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{r_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

One says that a Cartan scheme $\mathcal{C}$ is *connected*, if the group $\langle r_i \mid i \in I\rangle \subset \mathrm{Aut}(A)$ acts transitively on A, that is, if for all $a, b \in A$ with $a \neq b$ there exist $n \in \mathbb{N}_0$ and $i_1, i_2, \dots, i_n \in I$ such that $b = r_{i_n}\cdots r_{i_2}r_{i_1}(a)$. Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (r'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are called *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1\big(r_i(a)\big) = r'_{\varphi_0(i)}\big(\varphi_1(a)\big), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{3.3}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C}=\mathcal{C}(I,A,(r_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme. For all $i\in I$ and $a\in A$ define $\sigma_i^a\in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j)=\alpha_j-c_{ij}^a\alpha_i \quad \text{for all } j\in I. \tag{3.4}$$

This map is a reflection. The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C}))=A$ and the morphisms are generated by the maps $\sigma_i^a\in \mathrm{Hom}(a,r_i(a))$ with $i\in I$, $a\in A$. Formally, for $a,b\in A$ the set $\mathrm{Hom}(a,b)$ consists of the triples (b,f,a), where

$$f=\sigma_{i_n}^{r_{i_{n-1}}\cdots r_{i_1}(a)}\cdots\sigma_{i_2}^{r_{i_1}(a)}\sigma_{i_1}^a$$

and $b=r_{i_n}\cdots r_{i_2}r_{i_1}(a)$ for some $n\in\mathbb{N}_0$ and $i_1,\dots,i_n\in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3,f_2,a_2)\circ(a_2,f_1,a_1)=(a_3,f_2f_1,a_1)$$

for all $(a_3,f_2,a_2),(a_2,f_1,a_1)\in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. By abuse of notation one also writes $f\in\mathrm{Hom}(a,b)$ instead of $(b,f,a)\in\mathrm{Hom}(a,b)$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$. A Cartan scheme is called *connected* if its Weyl groupoid is connected.

Recall that a groupoid is a category such that all morphisms are isomorphisms. The Weyl groupoid $\mathcal{W}(\mathcal{C})$ of a Cartan scheme $\mathcal{C}$ is a groupoid, see [CH09]. For all $i\in I$ and $a\in A$ the inverse of σ_i^a is $\sigma_i^{r_i(a)}$. If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes, then $\mathcal{W}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C}')$ are isomorphic groupoids.

A groupoid G is called *connected*, if for each $a,b\in\mathrm{Ob}(G)$ the class $\mathrm{Hom}(a,b)$ is non-empty. Hence $\mathcal{W}(\mathcal{C})$ is a connected groupoid if and only if $\mathcal{C}$ is a connected Cartan scheme.

Definition 3.4. Let $\mathcal{C}=\mathcal{C}(I,A,(r_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme. For all $a\in A$ let $R^a\subset\mathbb{Z}^I$, and define $m_{i,j}^a=|R^a\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|$ for all $i,j\in I$ and $a\in A$. One says that

$$\mathcal{R}=\mathcal{R}\big(\mathcal{C},\big(R^a\big)_{a\in A}\big)$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a=R_+^a\cup -R_+^a$, where $R_+^a=R^a\cap\mathbb{N}_0^I$, for all $a\in A$.
(R2) $R^a\cap\mathbb{Z}\alpha_i=\{\alpha_i,-\alpha_i\}$ for all $i\in I$, $a\in A$.
(R3) $\sigma_i^a(R^a)=R^{r_i(a)}$ for all $i\in I$, $a\in A$.
(R4) If $i,j\in I$ and $a\in A$ such that $i\neq j$ and $m_{i,j}^a$ is finite, then $(r_ir_j)^{m_{i,j}^a}(a)=a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then $\mathcal{W}(\mathcal{R})=\mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected*, if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R}=\mathcal{R}(\mathcal{C},(R^a)_{a\in A})$ is a root system of type $\mathcal{C}$ and $\mathcal{R}'=\mathcal{R}'(\mathcal{C}',(R'^a_{a\in A'}))$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent*, if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0:I\to I'$, $\varphi_1:A\to A'$ as in Definition 3.3, and if the map $\varphi_0^*:\mathbb{Z}^I\to\mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i)=\alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a)=R'^{\varphi_1(a)}$ for all $a\in A$.

There exist many interesting examples of root systems of type $\mathcal{C}$ related to semisimple Lie algebras, Lie superalgebras and Nichols algebras of diagonal type, respectively. Further details and results can be found in [HY08] and [CH09].

Convention 3.5. In connection with Cartan schemes $\mathcal{C}$, upper indices usually refer to elements of A. Often, these indices will be omitted if they are uniquely determined by the context. The notation $w1_a$

and $1_a w'$ with $w, w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ and $a \in A$ means that $w \in \mathrm{Hom}(a, \_)$ and $w' \in \mathrm{Hom}(\_, a)$, respectively.

A fundamental result about Weyl groupoids is the following theorem.

Theorem 3.6. *(See [HY08, Theorem 1].) Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $\mathcal{W}$ be the abstract groupoid with $\mathrm{Ob}(\mathcal{W}) = A$ such that $\mathrm{Hom}(\mathcal{W})$ is generated by abstract morphisms $s_i^a \in \mathrm{Hom}(a, r_i(a))$, where $i \in I$ and $a \in A$, satisfying the relations*

$$s_i s_i 1_a = 1_a, \qquad (s_j s_k)^{m^a_{j,k}} 1_a = 1_a, \quad a \in A,\ i, j, k \in I,\ j \neq k,$$

see Convention 3.5. Here 1_a is the identity of the object a, and $(s_j s_k)^\infty 1_a$ is understood to be 1_a. The functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$, which is the identity on the objects, and on the set of morphisms is given by $s_i^a \mapsto \sigma_i^a$ for all $i \in I$, $a \in A$, is an isomorphism of groupoids.

Remark 3.7. Theorem 3.6 was formulated and proven in [HY08] using the language of semigroups. Translating the constructions there into category theory, the structures and arguments become very natural and more transparent. For example, the definition of the semigroup in [HY08, Definition 1] needs an artificial 0, which is not needed when dealing with morphisms of categories. (Idempotents e_a in [HY08, Definition 1] correspond to the identities of the objects, and elements in the semigroup with product 0 correspond to morphisms for which composition is not defined.) The (morphisms of the) groupoid $\mathcal{W}$ corresponds to the semigroup W in [HY08, Definition 1]. The existence of the functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$ means that the morphisms σ_i^a of $\mathcal{W}(\mathcal{R})$ satisfy the Coxeter relations. This was proven in [HY08, Proposition 1]. The functor is full by definition of $\mathcal{W}(\mathcal{R})$ and faithful by [HY08, Theorem 1].

If $\mathcal{C}$ is a Cartan scheme, then the Weyl groupoid $\mathcal{W}(\mathcal{C})$ admits a length function $\ell : \mathcal{W}(\mathcal{C}) \to \mathbb{N}_0$ such that

$$\ell(w) = \min\{k \in \mathbb{N}_0 \mid \exists i_1, \dots, i_k \in I,\ a \in A\colon\ w = \sigma_{i_1} \cdots \sigma_{i_k} 1_a\} \tag{3.5}$$

for all $w \in \mathcal{W}(\mathcal{C})$. If there exists a root system of type $\mathcal{C}$, then ℓ has very similar properties to the well-known length function for Weyl groups, see [HY08].

Lemma 3.8. *Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. Let $a \in A$. Then $-c^a_{ij} = \max\{m \in \mathbb{N}_0 \mid \alpha_j + m\alpha_i \in R^a_+\}$ for all $i, j \in I$ with $i \neq j$.*

Proof. By (C2) and (R3), $\sigma_i^{r_i(a)}(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \in R^a_+$. Hence $-c^a_{ij} \leqslant \max\{m \in \mathbb{N}_0 \mid \alpha_j + m\alpha_i \in R^a_+\}$. On the other hand, if $\alpha_j + m\alpha_i \in R^a_+$, then $\sigma_i^a(\alpha_j + m\alpha_i) = \alpha_j + (-c^a_{ij} - m)\alpha_i \in R^{r_i(a)}_+$ by (R3) and (R1), and hence $m \leqslant -c^a_{ij}$. This proves the lemma. □

Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. We say that $\mathcal{R}$ is *finite*, if R^a is finite for all $a \in A$. Following the terminology in [Kac90], for all $a \in A$ one defines

$$\left(R^a\right)^{\mathrm{re}} = \left\{w(\alpha_i) \;\middle|\; w \in \mathrm{Hom}(b, a),\ b \in A,\ i \in I\right\}, \tag{3.6}$$

the set of *real roots of* $a \in A$. Then $\mathcal{R}^{\mathrm{re}} = \mathcal{R}^{\mathrm{re}}(\mathcal{C}, ((R^a)^{\mathrm{re}})_{a\in A})$ is a root system of type $\mathcal{C}$ by [CH09, Proposition 2.9]. The following lemmata are well known for traditional root systems.

Lemma 3.9. *(See [CH09, Lemma 2.11].) Let $\mathcal{C}$ be a connected Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. The following are equivalent.*

(1) *$\mathcal{R}$ is finite.*
(2) *R^a is finite for at least one $a \in A$.*
(3) *$\mathcal{R}^{\mathrm{re}}$ is finite.*
(4) *$\mathcal{W}(\mathcal{R})$ is finite.*

Lemma 3.10. *(See [HY08, Corollary 5].) Let $\mathcal{C}$ be a connected Cartan scheme and $\mathcal{R}$ a finite root system of type $\mathcal{C}$. Then for all $a \in A$ there exist unique elements $b \in A$ and $w \in \mathrm{Hom}(b, a)$ such that $|R^a_+| = \ell(w) \geqslant \ell(w')$ for all $w' \in \mathrm{Hom}(b', a')$, $a', b' \in A$.*

3.3. The Weyl groupoid of a bicharacter

For an introduction to groupoids see [Bro87]. In this subsection the Weyl groupoid of a bicharacter is introduced following the general structure in the previous subsection. This definition, which differs from the original one in [Hec06, Section 5] and a related definition [AA08, Definition 3.3], has many advantages. One of them is that it fits better with the modern category theoretical point of view. Another one is that for quantized enveloping algebras of semisimple Lie algebras the Weyl groupoid becomes the Weyl group of the Lie algebra, see Example 3.13.

Let I be a non-empty finite set. Recall that a bicharacter on $\mathbb{Z}^I$ with values in $\Bbbk^\times$ is a map $\chi : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times$ such that

$$\chi(a+b, c) = \chi(a, c)\chi(b, c), \qquad \chi(c, a+b) = \chi(c, a)\chi(c, b) \tag{3.7}$$

for all $a, b, c \in \mathbb{Z}^I$. Then $\chi(0, a) = \chi(a, 0) = 1$ for all $a \in \mathbb{Z}^I$. Let $\mathcal{X}$ denote the set of bicharacters on $\mathbb{Z}^I$. For all $\chi \in \mathcal{X}$ the maps

$$\chi^{\mathrm{op}} : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \quad \chi^{\mathrm{op}}(a, b) = \chi(b, a), \tag{3.8}$$

$$\chi^{-1} : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \quad \chi^{-1}(a, b) = \chi(a, b)^{-1}, \tag{3.9}$$

and for all $\chi \in \mathcal{X}$, $w \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ the map

$$w^*\chi : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \quad w^*\chi(a, b) = \chi\big(w^{-1}(a), w^{-1}(b)\big) \tag{3.10}$$

are bicharacters on $\mathbb{Z}^I$. The equation

$$\big(ww'\big)^*\chi = w^*\big(w'^*\chi\big) \tag{3.11}$$

holds for all $w, w' \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ and all $\chi \in \mathcal{X}$.

Definition 3.11. Let $\chi \in \mathcal{X}$, $p \in I$, and $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. Then χ is called *p-finite*, if for all $j \in I \setminus \{p\}$ there exists $m \in \mathbb{N}_0$ such that $(m+1)_{q_{pp}} = 0$ or $q_{pp}^m q_{pj} q_{jp} = 1$.

Assume that χ is p-finite. Let $c^\chi_{pp} = 2$, and for all $j \in I \setminus \{p\}$ let

$$c^\chi_{pj} = -\min\big\{m \in \mathbb{N}_0 \bigm| (m+1)_{q_{pp}}\big(q_{pp}^m q_{pj} q_{jp} - 1\big) = 0\big\}.$$

If χ is i-finite for all $i \in I$, then the matrix $C^\chi = (c^\chi_{ij})_{i,j \in I}$ is called the *Cartan matrix* associated to χ. It is a generalized Cartan matrix, see Section 3.2.

For all $p \in I$ and $\chi \in \mathcal{X}$, where χ is p-finite, let $\sigma_p^\chi \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$,

$$\sigma_p^\chi(\alpha_j) = \alpha_j - c_{pj}^\chi \alpha_p \quad \text{for all } j \in I. \tag{3.12}$$

Towards the definition of the Weyl groupoid of a bicharacter, bijections $r_p : \mathcal{X} \to \mathcal{X}$ are defined for all $p \in I$. Namely, let

$$r_p : \mathcal{X} \to \mathcal{X}, \quad r_p(\chi) = \begin{cases} (\sigma_p^\chi)^* \chi & \text{if } \chi \text{ is } p\text{-finite,} \\ \chi & \text{otherwise.} \end{cases} \tag{3.13}$$

Let $p \in I$, $\chi \in \mathcal{X}$, $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. If χ is p-finite, then

$$\begin{aligned} r_p(\chi)(\alpha_p, \alpha_p) &= q_{pp}, & r_p(\chi)(\alpha_p, \alpha_j) &= q_{pj}^{-1} q_{pp}^{c_{pj}^\chi}, \\ r_p(\chi)(\alpha_i, \alpha_p) &= q_{ip}^{-1} q_{pp}^{c_{pi}^\chi}, & r_p(\chi)(\alpha_i, \alpha_j) &= q_{ij} q_{ip}^{-c_{pj}^\chi} q_{pj}^{-c_{pi}^\chi} q_{pp}^{c_{pi}^\chi c_{pj}^\chi} \end{aligned} \tag{3.14}$$

for all $i, j \in I$. It is a small exercise to check that then $(\sigma_p^\chi)^* \chi$ is p-finite, and

$$c_{pj}^{r_p(\chi)} = c_{pj}^\chi \quad \text{for all } j \in I, \qquad r_p^2(\chi) = \chi. \tag{3.15}$$

The bijections r_p, $p \in I$, generate a subgroup

$$\mathcal{G} = \langle r_p \mid p \in I \rangle$$

of the group of bijections of the set $\mathcal{X}$. For all $\chi \in \mathcal{X}$ let $\mathcal{G}(\chi)$ denote the $\mathcal{G}$-orbit of χ under the action of $\mathcal{G}$.

Let $\chi \in \mathcal{X}$ such that χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$. Then

$$\mathcal{C}(\chi) = \mathcal{C}\big(I, \mathcal{G}(\chi), (r_p)_{p \in I}, \big(C^{g(\chi)}\big)_{g \in \mathcal{G}}\big)$$

is a connected Cartan scheme by Eq. (3.15).

Definition 3.12. Let $\chi \in \mathcal{X}$ such that χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$. Then the Weyl groupoid of χ is the Weyl groupoid of the Cartan scheme $\mathcal{C}(\chi)$ and is denoted by $\mathcal{W}(\chi)$.

Clearly, $\mathcal{C}(\chi) = \mathcal{C}(\chi')$ and $\mathcal{W}(\chi) = \mathcal{W}(\chi')$ for all $\chi' \in \mathcal{G}(\chi)$.

Example 3.13. Let $C = (c_{ij})_{i,j \in I}$ be a generalized Cartan matrix. Let $\chi \in \mathcal{X}$, $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$, and assume that $q_{ii}^{c_{ij}} = q_{ij} q_{ji}$ for all $i, j \in I$, and that $(m+1)_{q_{ii}} \neq 0$ for all $i \in I$, $m \in \mathbb{N}_0$ with $m < \max\{-c_{ij} \mid j \in I\}$. (The latter is not an essential assumption, since if it fails, then one can replace C by another generalized Cartan matrix $\tilde{C}$, such that χ has this property with respect to $\tilde{C}$.) One says that χ is of *Cartan type*. Then χ is p-finite for all $p \in I$, and $c_{ij}^\chi = c_{ij}$ for all $i, j \in I$ by Definition 3.11. Eq. (3.14) gives that

$$\begin{gathered} r_p(\chi)(\alpha_i, \alpha_i) = q_{ii} = \chi(\alpha_i, \alpha_i), \\ r_p(\chi)(\alpha_i, \alpha_j) r_p(\chi)(\alpha_j, \alpha_i) = q_{ij} q_{ji} = r_p(\chi)(\alpha_i, \alpha_i)^{c_{ij}} \end{gathered}$$

for all $p, i, j \in I$. Hence $r_p(\chi)$ is again of Cartan type with the same Cartan matrix C. Thus χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$.

Assume now that C is a symmetrizable generalized Cartan matrix. For all $i \in I$ let $d_i \in \mathbb{N}$ such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. Let $q \in \Bbbk^\times$ such that $(m+1)_{q^{2d_i}} \neq 0$ for all $i \in I$ and $m \in \mathbb{N}_0$ with $m < \max\{-c_{ij} \mid j \in I\}$. Assume that $\chi(\alpha_i, \alpha_j) = q^{d_i c_{ij}}$ for all $i, j \in I$. Then χ is of Cartan type, hence χ is p-finite for all $p \in I$. Eq. (3.14) implies that $r_p(\chi) = \chi$ for all $p \in I$, and hence $\mathcal{G}(\chi)$ consists of precisely one element. In this case the Weyl groupoid $\mathcal{W}(\chi)$ is the group generated by the reflections σ_p^χ in Eq. (3.12), and hence $\mathcal{W}(\chi)$ is just the Weyl group associated to the generalized Cartan matrix C.

3.4. Roots and real roots

Let I be a non-empty finite set and let $\chi \in \mathcal{X}$, that is, a bicharacter on $\mathbb{Z}^I$ with values in $\Bbbk^\times$. Under suitable conditions there exists a canonical root system of type $\mathcal{C}(\chi)$ which is described in this subsection. It is based on the construction of a restricted PBW basis of Nichols algebras of diagonal type. More details can be found in Section 5 and in [AS02b] on Nichols algebras, in [Kha99] on the PBW basis, and in [Hec06, Section 3] on the root system.

Let $\Bbbk\mathbb{Z}^I$ denote the group algebra of $\mathbb{Z}^I$. Let $V \in {}^{\Bbbk\mathbb{Z}^I}_{\Bbbk\mathbb{Z}^I}\mathcal{YD}$ be an $|I|$-dimensional Yetter–Drinfel'd module of diagonal type. Let $\delta : V \to \Bbbk\mathbb{Z}^I \otimes V$ and $\cdot : \Bbbk\mathbb{Z}^I \otimes V \to V$ denote the left coaction and the left action of $\Bbbk\mathbb{Z}^I$ on V, respectively. Fix a basis $\{x_i \mid i \in I\}$ of V, elements g_i, where $i \in I$, and a matrix $(q_{ij})_{i,j\in I} \in (\Bbbk^\times)^{I\times I}$, such that

$$\delta(x_i) = g_i \otimes x_i, \qquad g_i \cdot x_j = q_{ij} x_j \quad \text{for all } i, j \in I.$$

Assume that $\chi(\alpha_i, \alpha_j) = q_{ij}$ for all $i, j \in I$. For all $\alpha \in \mathbb{Z}^I$ define

$$h^\chi(\alpha) = \begin{cases} \min\{m \in \mathbb{N} \mid (m)_{\chi(\alpha,\alpha)} = 0\} & \text{if } (m)_{\chi(\alpha,\alpha)} = 0, \text{ for some } m \in \mathbb{N}, \\ \infty & \text{otherwise.} \end{cases} \tag{3.16}$$

If $p \in I$ such that χ is p-finite, then

$$h^{r_p(\chi)}\big(\sigma_p^\chi(\alpha)\big) = h^\chi(\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^I, \tag{3.17}$$

by Eq. (3.10).

The tensor algebra $T(V)$ admits a universal braided Hopf algebra quotient $\mathfrak{B}(V)$, called the *Nichols algebra of* V. As an algebra, $\mathfrak{B}(V)$ has a unique $\mathbb{Z}^I$-grading

$$\mathfrak{B}(V) = \bigoplus_{\alpha \in \mathbb{Z}^I} \mathfrak{B}(V)_\alpha \tag{3.18}$$

such that $\deg x_i = \alpha_i$ for all $i \in I$, see [AHS08, Remark 2.8]. This is also a coalgebra grading. There exists a totally ordered index set $(L, \leqslant)$ and a family $(y_l)_{l\in L}$ of $\mathbb{Z}^I$-homogeneous elements $y_l \in \mathfrak{B}(V)$ such that the set

$$\begin{aligned} \big\{y_{l_1}^{m_1} y_{l_2}^{m_2} \cdots y_{l_k}^{m_k} \bigm| & k \geqslant 0,\ l_1, \dots, l_k \in L,\ l_1 > l_2 > \dots > l_k, \\ & m_i \in \mathbb{N}_0,\ m_i < h^\chi(\deg y_{l_i}) \text{ for all } i \in I\big\} \end{aligned} \tag{3.19}$$

forms a vector space basis of $\mathfrak{B}(V)$. Comparing dimensions of homogeneous components gives that the set

$$R_+^\chi = \{\deg y_l \mid l \in L\} \subset \mathbb{N}_0^I$$

depends on the matrix $(q_{ij})_{i,j\in I}$, but not on the choice of the basis $\{x_i \mid i \in I\}$, the set L, and the elements g_i, $i \in I$, and y_l, $l \in L$. (The proof of this uses Kharchenko's results [Kha99] on PBW bases of character Hopf algebras, and was discussed in [Hec06, Section 3].) Let

$$R^\chi = R^\chi_+ \cup -R^\chi_+.$$

Theorem 3.14. *Let $\chi \in \mathcal{X}$ such that χ' is p-finite for all $p \in I$, $\chi' \in \mathcal{G}(\chi)$. Then $\mathcal{R}(\chi) = \mathcal{R}(\mathcal{C}(\chi), (R^{\chi'})_{\chi' \in \mathcal{G}(\chi)})$ is a root system of type $\mathcal{C}(\chi)$.*

Proof. Axiom (R1) holds by definition. Next let us prove Axiom (R2). By definition of R^a, it suffices to consider the case $a = \chi$. Note that $\mathfrak{B}(V)_{m\alpha_i} = \Bbbk x_i^m$ for all $m \geqslant 0$, and x_i^m is zero if and only if $(m)^!_{q_{ii}} = 0$. Therefore by Eq. (3.19) and the definition of $h^\chi(\alpha_i)$ there is precisely one element y_l, where $l \in L$, of degree $m\alpha_i$, $m \geqslant 1$, and this is of degree α_i. This gives (R2).

Axiom (R3) holds by [Hec06, Proposition 1]. Finally, let $i, j \in I$ such that $i \neq j$ and m^χ_{ij} is finite. Since χ' is p-finite for all $p \in I$ and $\chi' \in \mathcal{G}(\chi)$, the calculation in the proof of [HY08, Lemma 5] — which does not use Axiom (R4) — implies that

$$(\sigma_i\sigma_j)^{m^\chi_{ij}} 1_\chi = \mathrm{id}.$$

Hence $(r_ir_j)^{m^\chi_{ij}}(\chi) = ((\sigma_i\sigma_j)^{m^\chi_{ij}} 1_\chi)^*\chi = \mathrm{id}^*\chi = \chi$. This proves the theorem. □

Remark 3.15. Let $\chi \in \mathcal{X}$. Assume that χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$. In general, the matrices $C^{\chi'}$ and C^χ, where $\chi' \in \mathcal{G}(\chi)$, do not coincide. If R^χ is finite, then $C^{\chi'}$ with $\chi' \in \mathcal{G}(\chi)$ does not need to be a Cartan matrix of finite type, see e.g. [Hec07, Table 1, row 17].

Lemma 3.16. *Let $\chi, \chi' \in \mathcal{X}$. Assume that χ'' is p-finite for all $p \in I$, $\chi'' \in \mathcal{G}(\chi) \cup \mathcal{G}(\chi')$.*

(i) *If $R^\chi_+ = R^{\chi'}_+$, then $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathrm{Hom}(\mathcal{W}(\chi))$.*
(ii) *Assume that R^χ_+ and $R^{\chi'}_+$ are finite sets. If $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathrm{Hom}(\mathcal{W}(\chi))$, then $R^\chi_+ = R^{\chi'}_+$.*

Proof. (i) Assume that $R^\chi_+ = R^{\chi'}_+$. Then $C^\chi = C^{\chi'}$ by Lemma 3.8. Therefore $\sigma^\chi_p = \sigma^{\chi'}_p$ in $\mathrm{Aut}(\mathbb{Z}^I)$ for all $p \in I$. Using the finiteness assumption on χ and χ' and Axiom (R3), by induction it follows that

$$\sigma_{i_1}\cdots\sigma^\chi_{i_k} = \sigma_{i_1}\cdots\sigma^{\chi'}_{i_k} \quad \text{in } \mathrm{Aut}\big(\mathbb{Z}^I\big), \qquad R^{(\sigma_{i_1}\cdots\sigma^\chi_{i_k})^*\chi} = R^{(\sigma_{i_1}\cdots\sigma^{\chi'}_{i_k})^*\chi'}, \tag{3.20}$$

and $C^{(\sigma_{i_1}\cdots\sigma^\chi_{i_k})^*\chi} = C^{(\sigma_{i_1}\cdots\sigma^{\chi'}_{i_k})^*\chi'}$ for all $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Hence $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathrm{Hom}(\mathcal{W}(\chi))$.

(ii) Since R^χ_+ is finite, $R^\chi = \{w^{-1}(\alpha_i) \mid w \in \mathrm{Hom}(\chi, \_) \subset \mathrm{Hom}(\mathcal{W}(\chi))\}$ by [CH09, Proposition 2.12]. By assumption on the Cartan matrices, the first formula in Eq. (3.20) holds for all $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Hence $R^\chi = R^{\chi'}$, and the lemma holds by (R1). □

Eqs. (3.8)–(3.10) describe natural relations between various bicharacters on $\mathbb{Z}^I$. These relations give rise to relations between different Weyl groupoids and root systems, respectively.

Proposition 3.17. *Let $\chi \in \mathcal{X}$.*

(a) *If χ is p-finite for all $p \in I$, then $C^{\chi^{\mathrm{op}}} = C^{\chi^{-1}} = C^\chi$.*
(b) *If χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$, then the Cartan schemes $\mathcal{C}(\chi)$ and $\mathcal{C}(\chi^{\mathrm{op}})$ are equivalent via $\varphi_0 = \mathrm{id}$ and $\varphi_1(\chi') = \chi'^{\mathrm{op}}$ for all $\chi' \in \mathcal{G}(\chi)$.*

(c) *If χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$, then the Cartan schemes $\mathcal{C}(\chi)$ and $\mathcal{C}(\chi^{-1})$ are equivalent via $\varphi_0 = \mathrm{id}$ and $\varphi_1(\chi') = \chi'^{-1}$ for all $\chi' \in \mathcal{G}(\chi)$.*
(d) *One has $R^{\chi^{\mathrm{op}}} = R^{\chi^{-1}} = R^{\chi}$.*

Proof. Part (a) follows immediately from Definition 3.11. Then the definition of $r_p(\chi)$ gives that the relations $\chi' \in \mathcal{G}(\chi)$, $\chi'^{\mathrm{op}} \in \mathcal{G}(\chi^{\mathrm{op}})$, and $\chi'^{-1} \in \mathcal{G}(\chi^{-1})$ are mutually equivalent, and hence (b) and (c) follow from (a). The relation $R_+^{\chi^{\mathrm{op}}} = R_+^{\chi}$ can be obtained by using twisting, see [AS02b, Proposition 3.9]. It remains to show that $R_+^{(\chi^{\mathrm{op}})^{-1}} = R_+^{\chi}$. For this recall that R_+^{χ} depends only on the dimensions of the $\mathbb{Z}^I$-homogeneous components of $\mathfrak{B}(V)$, and these depend only on the braiding c of V. The braiding and the inverse braiding of V define the same $\mathbb{Z}^I$-graded algebra quotient $\mathfrak{B}(V)$ of $T(V)$ (the quantum symmetrizer defined by c and its inverse, respectively, differ by an invertible factor corresponding to the permutation which reverses the words, and hence their kernels coincide). Let $\{x_i \mid i \in I\}$ be a basis of V as at the beginning of the subsection. Then the braiding of V and its inverse are given by $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $c^{-1}(x_i \otimes x_j) = q_{ji}^{-1} x_j \otimes x_i$ for all $i, j \in I$. Thus $(\chi^{\mathrm{op}})^{-1}$ is the bicharacter corresponding to the inverse of c, which yields the missing claim. $\square$

Later on we will need functions λ_i defined on the group of all bicharacters. In the next lemma these functions are defined and some of their properties are determined.

Lemma 3.18. *Let $\chi \in \mathcal{X}$, $p \in I$, and $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. Assume that χ is p-finite. Let $c_{pi} = c_{pi}^{\chi}$ for all $i \in I$. For all $i \in I \setminus \{p\}$ define*

$$\lambda_i(\chi) = (-c_{pi})^!_{q_{pp}} \prod_{s=0}^{-c_{pi}-1} \big(q_{pp}^s q_{pi} q_{ip} - 1\big).$$

Then for all $i \in I$ the following equations hold.

$$\lambda_i\big(r_p(\chi)\big) = \big(q_{pp}^{-c_{pi}} q_{pi} q_{ip}\big)^{c_{pi}} \lambda_i(\chi), \tag{3.21}$$

$$\lambda_i\big(\chi^{-1}\big) = \big(-q_{pp}^{-c_{pi}-1} q_{pi} q_{ip}\big)^{c_{pi}} \lambda_i(\chi). \tag{3.22}$$

Proof. Let $\bar{q}_{ij} = (r_p(\chi))(\alpha_i, \alpha_j)$ for all $i, j \in I$. Then by Eqs. (3.10) and (3.7) one gets

$$\bar{q}_{pp} = q_{pp}, \qquad \bar{q}_{ip}\bar{q}_{pi} = q_{ip}^{-1} q_{pi}^{-1} q_{pp}^{2c_{pi}} \quad \text{for all } i \in I \setminus \{p\}. \tag{3.23}$$

Eq. (3.22) follows easily from the definition of $\lambda_i(\chi)$ using the formulas $\chi^{-1}(\alpha_i, \alpha_j) = q_{ij}^{-1}$, where $i, j \in I$.

By Eq. (3.23) and Axiom (C2) one obtains that

$$\begin{aligned}\lambda_i\big(r_p(\chi)\big) &= (-c_{pi})^!_{\bar{q}_{pp}} \prod_{s=0}^{-c_{pi}-1} \big(\bar{q}_{pp}^s \bar{q}_{pi} \bar{q}_{ip} - 1\big) \\ &= (-c_{pi})^!_{q_{pp}} \prod_{s=0}^{-c_{pi}-1} \big(q_{pp}^{2c_{pi}+s} q_{pi}^{-1} q_{ip}^{-1} - 1\big).\end{aligned}$$

If $q_{pp}^{-c_{pi}} q_{ip} q_{pi} = 1$, then the latter formula is equal to $\lambda_i(\chi)$ and hence Eq. (3.21) holds. Otherwise, since χ is p-finite, one gets $q_{pp}^{1-c_{pi}} = 1$ and

$$\lambda_i\big(r_p(\chi)\big) = (-c_{pi})^!_{q_{pp}} \prod_{s=0}^{-c_{pi}-1} q_{pp}^{c_{pi}+s+1} q_{pi}^{-1} q_{ip}^{-1} \big(1 - q_{pp}^{-c_{pi}-1-s} q_{pi} q_{ip}\big)$$

$$= \big(q_{pp}^{-c_{pi}} q_{pi} q_{ip}\big)^{c_{pi}} q_{pp}^{(-c_{pi})(1-c_{pi})/2} (-c_{pi})^!_{q_{pp}} \prod_{s=0}^{-c_{pi}-1} \big(1 - q_{pp}^{s} q_{pi} q_{ip}\big).$$

By considering separately the cases where $-c_{pi}$ is even and odd, respectively, one can easily check that

$$(1 - c_{pi})_{q_{pp}} = 0, \qquad (-c_{pi})^!_{q_{pp}} \neq 0 \quad \Rightarrow \quad q_{pp}^{(-c_{pi})(1-c_{pi})/2} = (-1)^{c_{pi}}. \tag{3.24}$$

Hence Eq. (3.21) holds in the case $(1 - c_{pi})_{q_{pp}} = 0$, too. $\square$

4. A not so special Drinfel'd double

In this section the Drinfel'd double for a class of graded Hopf algebras is constructed and some properties are proven. In the literature, following Drinfeld's pioneering work [Dri87], various definitions of (multiparameter) quantizations of universal enveloping algebras of semisimple Lie algebras and Lie superalgebras appear. Often these doubles are quotients of a special case of the presented Drinfel'd double. Maybe the definitions most closest to those in the present paper are those in [KS08, Section 3], [Pei07], and [RS08b, Definition 1.5], which are more special, and the one in [RS08a, Sections 1.1, 8.1], which is more general. Our treatment, similarly to [RS08a], has the advantage that many combinatorial settings, mainly on the structure constants attached to some root systems, are removed, or they are shifted to assumptions on the Weyl groupoid.

4.1. Construction of the Drinfel'd double

The construction of a Drinfel'd double [Jos95, Section 3.2], also called quantum double [KS97, Section 8.2], is based on a skew-Hopf pairing of two Hopf algebras. We will follow this construction. Further, we will often work with the category ${}^{\Bbbk\mathbb{Z}^I}_{\Bbbk\mathbb{Z}^I}\mathcal{YD}$ of Yetter–Drinfel'd modules over the group algebra $\Bbbk\mathbb{Z}^I$ of $\mathbb{Z}^I$. Roughly speaking, the objects of this category are vector spaces equipped with a left action and left coaction of $\Bbbk\mathbb{Z}^I$ satisfying a compatibility condition, and morphisms are preserving both the left action and the left coaction. Precise definitions can be found e.g. in [Mon93, Section 10.6] and [AS02b, Section 1.2].

We keep the settings from the beginning of Section 3. Let I be a non-empty finite set and let $\chi \in \mathcal{X}$. Let $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. Let $\mathcal{U}^{+0} = \Bbbk[K_i, K_i^{-1} \mid i \in I]$ and $\mathcal{U}^{-0} = \Bbbk[L_i, L_i^{-1} \mid i \in I]$ be two copies of the group algebra of $\mathbb{Z}^I$. Let

$$V^+(\chi) \in {}^{\mathcal{U}^{+0}}_{\mathcal{U}^{+0}}\mathcal{YD}, \qquad V^-(\chi) \in {}^{\mathcal{U}^{-0}}_{\mathcal{U}^{-0}}\mathcal{YD} \tag{4.1}$$

be $|I|$-dimensional vector spaces over $\Bbbk$ with basis $\{E_i \mid i \in I\}$ and $\{F_i \mid i \in I\}$, respectively, such that the left action $\cdot$ and the left coaction δ of $\mathcal{U}^{+0}$ on $V^+(\chi)$ and of $\mathcal{U}^{-0}$ on $V^-(\chi)$, respectively, are determined by the formulas

$$K_i \cdot E_j = q_{ij} E_j, \qquad K_i^{-1} \cdot E_j = q_{ij}^{-1} E_j, \qquad \delta(E_i) = K_i \otimes E_i, \tag{4.2}$$

$$L_i \cdot F_j = q_{ji} F_j, \qquad L_i^{-1} \cdot F_j = q_{ji}^{-1} F_j, \qquad \delta(F_i) = L_i \otimes F_i \tag{4.3}$$

for all $i, j \in I$. Let

$$\mathcal{U}^+(\chi) = TV^+(\chi), \qquad \mathcal{U}^-(\chi) = TV^-(\chi) \tag{4.4}$$

denote the tensor algebra of $V^+(\chi)$ and $V^-(\chi)$, respectively. Since ${}^{\Bbbk\mathbb{Z}^I}_{\Bbbk\mathbb{Z}^I}\mathcal{YD}$ is a tensor category, the algebras $\mathcal{U}^+(\chi)$ and $\mathcal{U}^-(\chi)$ are Yetter–Drinfel'd modules over $\mathcal{U}^{+0}$ and $\mathcal{U}^{-0}$, respectively.

The main objects of study in this paper are the Drinfel'd double $D(\mathcal{V}^+(\chi), \mathcal{V}^-(\chi))$ of the Hopf algebras

$$\mathcal{V}^+(\chi) = \mathcal{U}^+(\chi)\,\#\,\mathcal{U}^{+0}, \qquad \mathcal{V}^-(\chi) = \big(\mathcal{U}^-(\chi)\,\#\,\mathcal{U}^{-0}\big)^{\mathrm{cop}} \tag{4.5}$$

and quotients of it. Here # denotes Radford's biproduct [Rad85], also called bosonization following Majid's terminology. As an algebra, it is a smash product, see [Mon93, Definition 4.1.3]. In particular,

$$K_iE_j = q_{ij}E_jK_i, \qquad L_iF_j = q_{ji}F_jL_i \tag{4.6}$$

for all $i, j \in I$, and the counits and coproducts of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ are determined by the equations

$$\begin{cases} \varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(L_i) = 1, \quad \varepsilon(F_i) = 0, \\ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(L_i) = L_i \otimes L_i, \\ \Delta\big(K_i^{-1}\big) = K_i^{-1} \otimes K_i^{-1}, \quad \Delta\big(L_i^{-1}\big) = L_i^{-1} \otimes L_i^{-1}, \\ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i \end{cases} \tag{4.7}$$

for all $i \in I$. The existence of the antipode follows from [Tak71].

The algebra $\mathcal{U}^+(\chi)$ itself is a braided Hopf algebra, see Proposition 4.1 below. A braided Hopf algebra is a Hopf algebra in a braided (for example Yetter–Drinfel'd) category. For further details we refer to [Tak00]. Moreover, under a connected Hopf algebra we mean a connected coalgebra in the sense of [Mon93, Definition 5.1.5].

Proposition 4.1. *(See [AS02b, Section 2.1].) The algebra $\mathcal{U}^+(\chi)$ is a connected braided Hopf algebra in the Yetter–Drinfel'd category ${}^{\mathcal{U}^{+0}}_{\mathcal{U}^{+0}}\mathcal{YD}$, where the left action and the left coaction of $\mathcal{U}^{+0}$ on $\mathcal{U}^+(\chi)$ are determined by the formulas*

$$K_i \cdot E_j = q_{ij}E_j, \qquad \delta(E_i) = K_i \otimes E_i \tag{4.8}$$

for $i, j \in I$. Further, the braiding $c \in \mathrm{Aut}_{\Bbbk}(\mathcal{U}^+(\chi) \otimes \mathcal{U}^+(\chi))$ is the canonical braiding of the category, that is

$$c\big(E \otimes E'\big) = E_{(-1)} \cdot E' \otimes E_{(0)}, \qquad c(E_i \otimes E_j) = q_{ij}E_j \otimes E_i \tag{4.9}$$

for all $i, j \in I$ and $E, E' \in \mathcal{U}^+(\chi)$. The braided coproduct $\underline{\Delta} : \mathcal{U}^+(\chi) \to \mathcal{U}^+(\chi) \otimes \mathcal{U}^+(\chi)$ is defined by

$$\underline{\Delta}(E_i) = E_i \otimes 1 + 1 \otimes E_i \quad \text{for all } i \in I. \tag{4.10}$$

Remark 4.2. The coproduct of $\mathcal{V}^+(\chi)$ and the braided coproduct of $\mathcal{U}^+(\chi)$ are related by the formula

$$\Delta(E) = E_{(1)} \otimes E_{(2)} = E^{(1)}\big(E^{(2)}\big)_{(-1)} \otimes \big(E^{(2)}\big)_{(0)} \quad \text{for all } E \in \mathcal{U}^+(\chi),$$

where $\underline{\Delta}(E) = E^{(1)} \otimes E^{(2)}$.

In order to form the Drinfel'd double $D(\mathcal{V}^+(\chi), \mathcal{V}^-(\chi))$, one needs a skew-Hopf pairing

$$\eta : \mathcal{V}^+(\chi) \times \mathcal{V}^-(\chi) \to \Bbbk, \quad (x, y) \mapsto \eta(x, y)$$

of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$. This means, see [Jos95, Section 3.2.1], that η is a bilinear map satisfying the equations

$$\eta(1, y) = \varepsilon(y), \qquad \eta(x, 1) = \varepsilon(x), \tag{4.11}$$

$$\eta\big(xx', y\big) = \eta\big(x', y_{(1)}\big)\eta\big(x, y_{(2)}\big), \qquad \eta\big(x, yy'\big) = \eta\big(x_{(1)}, y\big)\eta\big(x_{(2)}, y'\big), \tag{4.12}$$

$$\eta\big(S(x), y\big) = \eta\big(x, S^{-1}(y)\big) \tag{4.13}$$

for all $x, x' \in \mathcal{V}^+(\chi)$ and $y, y' \in \mathcal{V}^-(\chi)$. Equivalently, η is a Hopf pairing of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)^{\mathrm{cop}} = \mathcal{U}^-(\chi)\,\#\,\mathcal{U}^{-0}$. For all $\chi \in \mathcal{X}$ let us fix the skew-Hopf pairing given by the following proposition.

Proposition 4.3.

(i) *There exists a unique skew-Hopf pairing η of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ such that for all $i, j \in I$*

$$\eta(E_i, F_j) = -\delta_{i,j}, \qquad \eta(E_i, L_j) = 0, \qquad \eta(K_i, F_j) = 0, \qquad \eta(K_i, L_j) = q_{ij}.$$

(ii) *The skew-Hopf pairing η satisfies the equations*

$$\eta(EK, FL) = \eta(E, F)\eta(K, L)$$

for all $E \in \mathcal{U}^+(\chi)$, $F \in \mathcal{U}^-(\chi)$, $K \in \mathcal{U}^{+0}$, and $L \in \mathcal{U}^{-0}$.

Proof. (i) First we prove the uniqueness of the pairing. Since $\mathcal{V}^+(\chi)$ is generated by the set $\{E_i, K_i, K_i^{-1} \mid i \in I\}$, the linearity of η in the first argument and the first formula in Eq. (4.12) tell that η is determined by the values $\eta(x, y)$, where

$$x \in \{1\} \cup \big\{K_i, K_i^{-1}, E_i \;\big|\; i \in I\big\} \tag{4.14}$$

and $y \in \mathcal{V}^-(\chi)$. Since Δ maps the elements of the latter set to linear combinations of tensor products of the same elements, see Eq. (4.7), the linearity of η in the second argument and the second formula in Eq. (4.12) yield that η is determined by the values $\eta(x, y)$, where x is as in Rel. (4.14) and

$$y \in \{1\} \cup \big\{L_i, L_i^{-1}, F_i \;\big|\; i \in I\big\}. \tag{4.15}$$

Further, by Eq. (4.11) and relations $K_iK_i^{-1} = 1$ and $L_iL_i^{-1} = 1$ for all $i \in I$ it suffices to consider the case

$$x \in \{K_i, E_i \mid i \in I\}, \qquad y \in \{L_i, F_i \mid i \in I\}.$$

The numbers $\eta(x, y)$ for such x, y are given in the proposition.

The existence of the pairing can be proven for example by following the method in [KS97, Proposition 6.34]. For all $i \in I$ let $\bar{K}_i, \bar{E}_i \in (\mathcal{V}^-(\chi))^*$ be such that

$$\bar{K}_i(FL) = \varepsilon(F)\prod_{k\in I} q_{ik}^{m_k}, \qquad \bar{E}_i(F_{j_1}\cdots F_{j_l}L) = -\delta_{l,1}\delta_{i,j_1} \tag{4.16}$$

for all $F \in \mathcal{U}^-(\chi)$, $l \in \mathbb{N}_0$, $j_1, \ldots, j_l \in I$, $L = \prod_{k \in I} L_k^{m_k}$, $m_k \in \mathbb{Z}$ for all $k \in I$. One checks that

$$\bar{K}_i(ab) = \bar{K}_i(a)\bar{K}_i(b), \qquad \bar{E}_i(ab) = \bar{E}_i(a)\varepsilon(b) + \bar{K}_i(a)\bar{E}_i(b)$$

for all $i \in I$ and $a, b \in \mathcal{V}^-(\chi)$. Regard the dual space $\mathcal{V}^-(\chi)^*$ as an algebra with convolution product: $(fg)(a) = f(a_{(1)})g(a_{(2)})$ for all $a \in \mathcal{V}^-(\chi)$, $f, g \in \mathcal{V}^-(\chi)^*$. Then for all $i \in I$ the functionals $\bar{K}_i$ are invertible, and the relations

$$\bar{K}_i\bar{K}_j = \bar{K}_j\bar{K}_i, \qquad q_{ij}\bar{K}_i\bar{E}_j = \bar{E}_j\bar{K}_i$$

hold for all $i, j \in I$. Thus there exists a unique algebra antihomomorphism $\gamma : \mathcal{V}^+(\chi) \to \mathcal{V}^-(\chi)^*$ with $\gamma(K_i) = \bar{K}_i$ and $\gamma(E_i) = \bar{E}_i$ for all $i \in I$. Let $\eta(a, b) = \gamma(a)(b)$ for all $a \in \mathcal{V}^+(\chi)$, $b \in \mathcal{V}^-(\chi)$. Then the first equations in (4.11) and (4.12) hold for all $y \in \mathcal{V}^-(\chi)$ by definition of η and γ. The second equations in the same lines can be proven for all $x \in \mathcal{V}^+(\chi)$ by induction on the number of factors of x.

Finally, one has to check that Eq. (4.13) holds for all x, y. Using the fact that S and S^{-1} are algebra and coalgebra antihomomorphisms, one can reduce the problem to the case when x and y are generators. Again in this case the equation can be easily shown.

(ii) Let $E \in \mathcal{U}^+(\chi)$, $F \in \mathcal{U}^-(\chi)$, $K \in \mathcal{U}^{+0}$, and $L \in \mathcal{U}^{-0}$. By the definition of the coproduct of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ one obtains the following equations.

$$\begin{aligned} E_{(1)}K_{(1)}\eta(E_{(2)}K_{(2)}, L) &= EK_{(1)}\eta(K_{(2)}, L), \\ \eta(EK, F) &= \eta(K, F_{(1)})\eta(E, F_{(2)}) = \varepsilon(K)\eta(E, F), \\ \eta(EK, FL) &= \eta(E_{(1)}K_{(1)}, F)\eta(E_{(2)}K_{(2)}, L) \\ &= \eta(EK_{(1)}, F)\eta(K_{(2)}, L) \\ &= \varepsilon(K_{(1)})\eta(E, F)\eta(K_{(2)}, L) = \eta(E, F)\eta(K, L). \end{aligned}$$

This proves the proposition. □

Remark 4.4. One can slightly generalize Proposition 4.3. Let $(a_i)_{i \in I} \in \mathbb{k}^I$. The proof of the proposition shows that if one replaces equation $\eta(E_i, F_j) = -\delta_{i,j}$ by $\eta(E_i, F_j) = a_i\delta_{i,j}$, then the pairing η still exists and is unique. In what follows, we will stick to the setting in Proposition 4.3.

The following definition is a combination of Proposition 4.3 and the definition in [Jos95, Section 3.2.4].

Definition 4.5. Let $\chi \in \mathcal{X}$. For all $i, j \in I$ let $q_{ij} = \chi(\alpha_i, \alpha_j)$. Let $\mathcal{U}(\chi)$ be the Drinfel'd double of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ with respect to the skew-Hopf pairing in Proposition 4.3, that is $\mathcal{U}(\chi)$ is the unique Hopf algebra such that

(1) $\mathcal{U}(\chi) = \mathcal{V}^+(\chi) \otimes \mathcal{V}^-(\chi)$ as a vector space,
(2) the maps $\mathcal{V}^+(\chi) \to \mathcal{U}(\chi)$, $x \mapsto x \otimes 1$ and $\mathcal{V}^-(\chi) \to \mathcal{U}(\chi)$, $y \mapsto 1 \otimes y$ are Hopf algebra maps,
(3) the product of $\mathcal{U}(\chi)$ is given by

$$(x \otimes y)(x' \otimes y') = x\eta(x'_{(1)}, S(y_{(1)}))x'_{(2)} \otimes y_{(2)}\eta(x'_{(3)}, y_{(3)})y' \tag{4.17}$$

for all $x, x' \in \mathcal{V}^+(\chi)$ and $y, y' \in \mathcal{V}^-(\chi)$.

In what follows, the tensor product sign in elements of $\mathcal{U}(\chi)$ will be omitted. Let $\mathcal{U}^0(\chi)$ denote the commutative cocommutative Hopf subalgebra

$$\mathcal{U}^0(\chi)=\Bbbk\big[K_i, K_i^{-1}, L_i, L_i^{-1} \bigm| i\in I\big] \tag{4.18}$$

of $\mathcal{U}(\chi)$. For all $\mu=\sum_{i\in I} m_i\alpha_i\in\mathbb{Z}^I$ let

$$K_\mu=\prod_{i\in I} K_i^{m_i}, \qquad L_\mu=\prod_{i\in I} L_i^{m_i}.$$

Alternatively, one can define the algebra $\mathcal{U}(\chi)$ in terms of generators and relations. The equivalence of these definitions is an easy standard calculation, see for example [Jos95, Lemma 3.2.5].

Proposition 4.6. *The algebra $\mathcal{U}(\chi)$ is generated by the elements K_i, K_i^{-1}, L_i, L_i^{-1}, E_i, and F_i, where $i\in I$, and defined by the relations*

$$XY=YX \quad \text{for all } X, Y\in\big\{K_i, K_i^{-1}, L_i, L_i^{-1} \bigm| i\in I\big\}, \tag{4.19}$$

$$K_iK_i^{-1}=1, \qquad L_iL_i^{-1}=1, \tag{4.20}$$

$$K_iE_jK_i^{-1}=q_{ij}E_j, \qquad L_iE_jL_i^{-1}=q_{ji}^{-1}E_j, \tag{4.21}$$

$$K_iF_jK_i^{-1}=q_{ij}^{-1}F_j, \qquad L_iF_jL_i^{-1}=q_{ji}F_j, \tag{4.22}$$

$$E_iF_j-F_jE_i=\delta_{i,j}(K_i-L_i). \tag{4.23}$$

Note that by definition the coalgebra structure of $\mathcal{U}(\chi)$ is determined by Eqs. (4.7).

Remark 4.7. 1. Assume that there exist a symmetrizable generalized Cartan matrix $C=(c_{ij})_{i,j\in I}$ with integer entries, positive integers d_i, $i\in I$, and a number $q\in\Bbbk^\times$ which is not a root of 1, such that

$$q_{ij}=q^{d_ic_{ij}} \quad \text{for all } i, j\in I.$$

Then the quantized symmetrizable Kac–Moody algebra associated to the matrix C, see [Jos95, Definition 3.2.9], [Lus93, 3.1.1] and Remark 5.7, is a quotient of the algebra $\mathcal{U}(\chi)$ by a Hopf ideal. In the special case when C is of finite type, the quantized Kac–Moody algebra is the Drinfel'd–Jimbo algebra or quantized enveloping algebra of the semisimple Lie algebra corresponding to C. See also Remark 5.7 and Theorem 5.8.

2. Usually, on the right-hand side of Eq. (4.23) a denominator appears. This allows an easier consideration of classical limits and specialization arguments. In our paper we will neither consider classical limits, nor will use specialization. Omitting the denominator we even achieve a slight generalization of the traditional setting by admitting the cases when $q_{ii}=\pm1$ for some $i\in I$.

3. Quantized Lie superalgebras, see [KT91, Definition 2.1], and quantized enveloping algebras for Borcherds superalgebras, see [BKM98], are quotients of algebras of the form $\mathcal{U}(\chi)$ or $\mathcal{U}(\chi)\,\#\,\Bbbk\Gamma$, too, where Γ is a finite group and # denotes Radford's biproduct, and $\chi=\chi^{\mathrm{op}}$ again has to satisfy some additional conditions depending on the underlying Lie superalgebra.

4. Two-parameter quantum groups, see e.g. [BW04] and [BGH06], are quotients of algebras of the form $\mathcal{U}(\chi)$, where the definition of χ needs two parameters. In these examples $\chi\neq\chi^{\mathrm{op}}$.

Remark 4.8. By Eqs. (4.4) and (4.21) the vector space $V^+(\chi)$ and the algebra $\mathcal{U}^+(\chi)$ are Yetter–Drinfel'd modules over $\mathcal{U}^0(\chi)$.

The algebra $\mathcal{U}(\chi)$ admits a unique $\mathbb{Z}^I$-grading

$$\mathcal{U}(\chi)=\bigoplus_{\mu\in\mathbb{Z}^I}\mathcal{U}(\chi)_\mu,$$
$$1\in\mathcal{U}(\chi)_0,\qquad \mathcal{U}(\chi)_\mu\mathcal{U}(\chi)_\nu\subset\mathcal{U}(\chi)_{\mu+\nu}\quad\text{for all } \mu,\nu\in\mathbb{Z}^I, \tag{4.24}$$

such that $K_i, K_i^{-1}, L_i, L_i^{-1}\in\mathcal{U}(\chi)_0$, $E_i\in\mathcal{U}(\chi)_{\alpha_i}$, and $F_i\in\mathcal{U}(\chi)_{-\alpha_i}$ for all $i\in I$. For all $\mu=\sum_{i\in I}m_i\alpha_i\in\mathbb{Z}^I$ let $|\mu|=\sum_{i\in I}m_i\in\mathbb{Z}$. The decomposition

$$\mathcal{U}(\chi)=\bigoplus_{m\in\mathbb{Z}}\mathcal{U}(\chi)_m,\quad \text{where } \mathcal{U}(\chi)_m=\bigoplus_{\mu\in\mathbb{Z}^I:\ |\mu|=m}\mathcal{U}(\chi)_\mu, \tag{4.25}$$

gives a $\mathbb{Z}$-grading of $\mathcal{U}(\chi)$ called the *standard grading*. For any $\mathbb{Z}^I$-homogeneous subquotient $\mathcal{U}'$ of $\mathcal{U}(\chi)$ the notation $\mathcal{U}'_\alpha$ and $\mathcal{U}'_m$ for the homogeneous components of degree $\alpha\in\mathbb{Z}^I$ and $m\in\mathbb{Z}$, respectively, will be used. Note that in general the subspaces $\mathcal{U}'_0$ for $0\in\mathbb{Z}^I$ and $\mathcal{U}'_0$ for $0\in\mathbb{Z}$ are different.

Proposition 4.9. *Let $\chi\in\mathcal{X}$.*

(1) *Let $\underline{a}=(a_i)_{i\in I}\in(\Bbbk^\times)^I$. There exists a unique algebra automorphism $\varphi_{\underline{a}}$ of $\mathcal{U}(\chi)$ such that*

$$\varphi_{\underline{a}}(K_i)=K_i,\qquad \varphi_{\underline{a}}(L_i)=L_i,\qquad \varphi_{\underline{a}}(E_i)=a_iE_i,\qquad \varphi_{\underline{a}}(F_i)=a_i^{-1}F_i. \tag{4.26}$$

(2) *Let τ be a permutation of I and let $\hat\tau$ be the automorphism of $\mathbb{Z}^I$ given by $\hat\tau(\alpha_i)=\alpha_{\tau(i)}$ for all $i\in I$. Then there exists a unique algebra isomorphism $\varphi_\tau:\mathcal{U}(\chi)\to\mathcal{U}(\hat\tau^*\chi)$ such that*

$$\varphi_\tau(K_i)=K_{\tau(i)},\qquad \varphi_\tau(L_i)=L_{\tau(i)},$$
$$\varphi_\tau(E_i)=E_{\tau(i)},\qquad \varphi_\tau(F_i)=F_{\tau(i)}. \tag{4.27}$$

(3) *For all $m\in\mathbb{Z}$ there exists a unique algebra automorphism φ_m of $\mathcal{U}(\chi)$ such that*

$$\varphi_m(K_i)=K_i,\qquad \varphi_m(L_i)=L_i,$$
$$\varphi_m(E_i)=K_i^mL_i^{-m}E_i,\qquad \varphi_m(F_i)=F_iK_i^{-m}L_i^m. \tag{4.28}$$

(4) *There exists a unique algebra automorphism ϕ_1 of $\mathcal{U}(\chi)$ such that*

$$\phi_1(K_i)=K_i^{-1},\qquad \phi_1(L_i)=L_i^{-1},$$
$$\phi_1(E_i)=F_iL_i^{-1},\qquad \phi_1(F_i)=K_i^{-1}E_i. \tag{4.29}$$

(5) *There is a unique algebra isomorphism $\phi_2:\mathcal{U}(\chi)\to\mathcal{U}(\chi^{-1})$ such that*

$$\phi_2(K_i)=K_i,\qquad \phi_2(L_i)=L_i,\qquad \phi_2(E_i)=F_i,\qquad \phi_2(F_i)=-E_i. \tag{4.30}$$

(6) *The algebra map $\phi_3 : \mathcal{U}(\chi) \to \mathcal{U}(\chi^{\mathrm{op}})^{\mathrm{cop}}$ defined by the formulas*

$$\phi_3(K_i) = L_i, \qquad \phi_3(L_i) = K_i, \qquad \phi_3(E_i) = F_i, \qquad \phi_3(F_i) = E_i \tag{4.31}$$

is an isomorphism of Hopf algebras.

(7) *There is a unique algebra antiautomorphism ϕ_4 of $\mathcal{U}(\chi)$ such that*

$$\phi_4(K_i) = K_i, \qquad \phi_4(L_i) = L_i, \qquad \phi_4(E_i) = F_i, \qquad \phi_4(F_i) = E_i. \tag{4.32}$$

Proof. One has to check the compatibility of the definitions with the defining relations of $\mathcal{U}(\chi)$, which is easy. The bijectivity can be proven by writing down the inverse map explicitly, see also Proposition 4.12 below. In case of the map ϕ_3 note that one has $\Delta(\phi_3(X)) = \phi_3(X_{(2)}) \otimes \phi_3(X_{(1)})$ for all generators X of $\mathcal{U}(\chi)$ which implies that ϕ_3 is a coalgebra antihomomorphism. □

Corollary 4.10. *The antipode of $\mathcal{U}(\chi)$ can be obtained as $S = \phi_1\phi_4\varphi_{\underline{a}}$, where $a_i = -1$ for all $i \in I$.*

Proof. Eqs. (4.7) imply that

$$\begin{gathered} S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iL_i^{-1}, \\ S(K_i) = K_i^{-1}, \qquad S(L_i) = L_i^{-1} \end{gathered} \tag{4.33}$$

for all $i \in I$. Then it is easy to check that the equation $S = \phi_1\phi_4\varphi_{\underline{a}}$ holds on the generators of $\mathcal{U}(\chi)$. Thus the corollary follows since both sides of the equation are algebra antihomomorphisms. □

The description of φ_1 below will be used in the proof of Lemma 5.5.

Lemma 4.11. *Let $\underline{a} \in (\Bbbk^\times)^I$ with $a_i = q_{ii}^{-1}$ for all $i \in I$. Then*

$$\varphi_1\varphi_{\underline{a}}(E) = \chi(\mu,\mu)EK_\mu L_\mu^{-1}$$

for all $\mu \in \mathbb{Z}^I$ and all $E \in \mathcal{U}(\chi)_\mu$.

Proof. Check that the map $\mathcal{U}^+(\chi) \to \mathcal{U}(\chi)$, $E \mapsto \chi(\mu,\mu)EK_\mu L_\mu^{-1}$ for all $\mu \in \mathbb{Z}^I$ and $E \in \mathcal{U}^+(\chi)_\mu$, is an algebra homomorphism. On the generators it coincides with the restriction of $\varphi_1\varphi_{\underline{a}}$ to $\mathcal{U}^+(\chi)$. □

Proposition 4.12. *The isomorphisms in Proposition 4.9 satisfy the following relations.*

(i) *Let $\underline{a}, \underline{b} \in (\Bbbk^\times)^I$ and $m, n \in \mathbb{Z}$. Then $\varphi_{\underline{a}}\varphi_{\underline{b}} = \varphi_{\underline{c}}$, $\varphi_m\varphi_{\underline{a}} = \varphi_{\underline{a}}\varphi_m$, and $\varphi_m\varphi_n = \varphi_{m+n}$, where $c_i = a_ib_i$ for all $i \in I$.*

(ii) *Let $\underline{a} \in (\Bbbk^\times)^I$ and $t \in \{1,2,3,4\}$. Then $\varphi_{\underline{a}}\phi_t = \phi_t\varphi_{\underline{b}}$, where $b_i = a_i^{-1}$ for all $i \in I$.*

(iii) *Let $m \in \mathbb{Z}$ and $\underline{a} \in (\Bbbk^\times)^I$ with $a_i = q_{ii}^{-2m}$ for all $i \in I$. Then $\varphi_m\phi_1 = \phi_1\varphi_m\varphi_{\underline{a}}$, $\varphi_m\phi_2 = \phi_2\varphi_{-m}\varphi_{\underline{a}}^{-1}$, $\varphi_m\phi_3 = \phi_3\varphi_m\varphi_{\underline{a}}$, and $\varphi_m\phi_4 = \phi_4\varphi_{-m}$.*

(iv) *Let $\underline{a}, \underline{b} \in (\Bbbk^\times)^I$ with $a_i = q_{ii}$ and $b_i = -1$ for all $i \in I$. Then $\phi_1^2 = \varphi_{-1}\varphi_{\underline{a}}$, $\phi_2^2 = \varphi_{\underline{b}}$, and $\phi_3^2 = \phi_4^2 = \mathrm{id}$.*

(v) *Let $\underline{a}, \underline{b} \in (\Bbbk^\times)^I$ with $a_i = q_{ii}^{-1}$ and $b_i = -1$ for all $i \in I$. Then $\phi_1\phi_2 = \phi_2\phi_1\varphi_1\varphi_{\underline{a}}\varphi_{\underline{b}}$, $\phi_1\phi_3 = \phi_3\phi_1\varphi_{\underline{a}}$, and $\phi_1\phi_4 = \phi_4\phi_1\varphi_1\varphi_{\underline{a}}^2$.*

(vi) *Let $\underline{b} \in (\Bbbk^\times)^I$ with $b_i = -1$ for all $i \in I$. Then $\phi_2\phi_3 = \phi_3\phi_2\varphi_{\underline{b}}$, $\phi_2\phi_4 = \phi_4\phi_2\varphi_{\underline{b}}$, and $\phi_3\phi_4 = \phi_4\phi_3$.*

Proof. Evaluate both sides of the equations on the generators of $\mathcal{U}(\chi)$ and compare the results. $\square$

For arbitrary $X, Y \in \mathcal{U}(\chi)$ and $K \in \mathcal{U}^0(\chi)$ let

$$[X, Y] = XY - YX, \qquad K \cdot X := (\operatorname{ad} K)X = K_{(1)} X S(K_{(2)}),$$

where ad denotes left adjoint action. This interpretation of the operation $\cdot$ is consistent with Rels. (4.2), (4.3), (4.21) and (4.22). For the computation of commutation relations in $\mathcal{U}(\chi)$ later on the following lemma will be useful. The proof is a direct consequence of Proposition 4.6.

Lemma 4.13. *Let $p \in I$ and $X \in \mathcal{U}(\chi)$. Then*

$$\left[K_p^{-1} E_p, X\right] = K_p^{-1}\left(E_p X - (K_p \cdot X) E_p\right) = K_p^{-1} (\operatorname{ad} E_p) X, \tag{4.34}$$

$$\left[X, F_p L_p^{-1}\right] = \left(X F_p - F_p\left(L_p^{-1} \cdot X\right)\right) L_p^{-1}. \tag{4.35}$$

The next claim is known as the *triangular decomposition of $\mathcal{U}(\chi)$*.

Proposition 4.14. *The multiplication maps*

$$\mathrm{m} : \mathcal{U}^+(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi) \to \mathcal{U}(\chi),$$
$$\mathrm{m} : \mathcal{U}^-(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^+(\chi) \to \mathcal{U}(\chi)$$

are isomorphisms of $\mathbb{Z}^I$-graded vector spaces.

Proof. The first map is an isomorphism by construction of $\mathcal{U}(\chi)$. The proof for the second one is also standard. It relies mainly on the fact that Eq. (4.17) has an "inverse" which tells that

$$xy = \eta(x_{(1)}, y_{(1)}) y_{(2)} x_{(2)} \eta\left(x_{(3)}, S(y_{(3)})\right)$$

for all $x \in \mathcal{V}^+(\chi)$ and $y \in \mathcal{V}^-(\chi)$. $\square$

4.2. Kashiwara maps

For quantized enveloping algebras $U_q(\mathfrak{g})$ of semisimple Lie algebras $\mathfrak{g}$ Kashiwara [Kas91] constructed certain skew-derivations of the upper triangular part $U_q^+(\mathfrak{g})$ by considering commutators in $U_q(\mathfrak{g})$. This construction can be generalized to our setting.

Lemma 4.15. *For all $i \in I$ there exist unique linear maps $\partial_i^K, \partial_i^L \in \operatorname{End}_{\Bbbk}(\mathcal{U}^+(\chi))$ such that*

$$[E, F_i] = \partial_i^K(E) K_i - L_i \partial_i^L(E) \quad \text{for all } E \in \mathcal{U}^+(\chi).$$

The maps $\partial_i^K, \partial_i^L \in \operatorname{End}_{\Bbbk}(\mathcal{U}^+(\chi))$ are skew-derivations. More precisely,

$$\partial_i^K(1) = \partial_i^L(1) = 0, \qquad \partial_i^K(E_j) = \partial_i^L(E_j) = \delta_{i,j}, \tag{4.36}$$

$$\partial_i^K\left(EE'\right) = \partial_i^K(E)\left(K_i \cdot E'\right) + E\partial_i^K\left(E'\right),$$

$$\partial_i^L\left(EE'\right) = \partial_i^L(E)E' + \left(L_i^{-1} \cdot E\right)\partial_i^L\left(E'\right) \tag{4.37}$$

for all $i, j \in I$ and $E, E' \in \mathcal{U}^+(\chi)$.

Proof. The triangular decomposition of $\mathcal{U}(\chi)$ and Rels. (4.21) imply uniqueness of the maps ∂_i^K and ∂_i^L. Since $\mathcal{U}^+(\chi)$ is the free algebra generated by $V^+(\chi)$, the existence of the maps ∂_i^K and ∂_i^L follows from Rels. (4.23) and the formula

$$\begin{aligned}\left[EE', F_i\right] &= [E, F_i]E' + E\left[E', F_i\right]\\ &= \left(\partial_i^K(E)K_i - L_i\partial_i^L(E)\right)E' + E\left(\partial_i^K\left(E'\right)K_i - L_i\partial_i^L\left(E'\right)\right)\\ &= \left(\partial_i^K(E)\left(K_i \cdot E'\right) + E\partial_i^K\left(E'\right)\right)K_i\\ &\quad - L_i\left(\partial_i^L(E)E' + \left(L_i^{-1}\cdot E\right)\partial_i^L\left(E'\right)\right),\end{aligned}$$

where $E, E' \in \mathcal{U}^+(\chi)$. This also proves the last part of the lemma. □

The maps ∂_i^K, ∂_i^L with $i \in I$ are variations of Lusztig's maps r_i and ${}_ir$ in [Lus93], as the second part of Lemma 4.15 shows.

Lemma 4.16. *Let $i, j \in I$ and $E \in \mathcal{U}^+(\chi)$. Then*

$$\partial_i^K(K_j \cdot E) = q_{ji}K_j \cdot \left(\partial_i^K(E)\right), \qquad \partial_i^K(L_j \cdot E) = q_{ij}^{-1}L_j \cdot \left(\partial_i^K(E)\right), \tag{4.38}$$

$$\partial_i^L(K_j \cdot E) = q_{ji}K_j \cdot \left(\partial_i^L(E)\right), \qquad \partial_i^L(L_j \cdot E) = q_{ij}^{-1}L_j \cdot \left(\partial_i^L(E)\right), \tag{4.39}$$

$$\partial_i^K\partial_j^L = \partial_j^L\partial_i^K. \tag{4.40}$$

Proof. The first equation in (4.38) holds for $E = 1$ and $E = E_m$, where $m \in I$, by Eqs. (4.36) and (4.21). Further, for $E, E' \in \mathcal{U}^+(\chi)$ one gets

$$\begin{aligned}\partial_i^K\left(K_j \cdot \left(EE'\right)\right) &= \partial_i^K\left((K_j \cdot E)\left(K_j \cdot E'\right)\right)\\ &= \partial_i^K(K_j \cdot E)\left(K_iK_j \cdot E'\right) + (K_j \cdot E)\partial_i^K\left(K_j \cdot E'\right).\end{aligned}$$

Thus the first equation in (4.38) follows by induction on the $\mathbb{Z}$-degree of E using Eq. (4.19). The second equation in (4.38) and the equations in (4.39) can be obtained similarly.

Now we prove Eq. (4.40). Using Eq. (4.36) one obtains that

$$\partial_i^K\partial_j^L(E) = \partial_j^L\partial_i^K(E) = 0$$

for all $i, j \in I$ and $E \in \{1, E_m \mid m \in I\}$. Further,

$$\begin{aligned}&\left(\partial_i^K\partial_j^L - \partial_j^L\partial_i^K\right)\left(EE'\right)\\ &\quad = \partial_i^K\left(\partial_j^L(E)E' + \left(L_j^{-1}\cdot E\right)\partial_j^L\left(E'\right)\right) - \partial_j^L\left(\partial_i^K(E)\left(K_i \cdot E'\right) + E\partial_i^K\left(E'\right)\right)\\ &\quad = \left(\partial_i^K\partial_j^L - \partial_j^L\partial_i^K\right)(E)\left(K_i \cdot E'\right) + \left(L_j^{-1}\cdot E\right)\left(\partial_i^K\partial_j^L - \partial_j^L\partial_i^K\right)\left(E'\right)\end{aligned}$$

for all $E, E' \in \mathcal{U}^+(\chi)$ because of the first part of the lemma. Thus the claim follows by induction. □

The following statement gives a characterization of a class of ideals of $\mathcal{U}(\chi)$ compatible with the triangular decomposition of $\mathcal{U}(\chi)$. This proposition seems to be new even for multiparameter quantizations of Kac–Moody algebras, see [KS08, Proposition 3.4].

Proposition 4.17. *Let $\mathcal{I}^+ \subset \mathcal{U}^+(\chi) \cap \ker \varepsilon$ and $\mathcal{I}^- \subset \mathcal{U}^-(\chi) \cap \ker \varepsilon$ be a (not necessarily $\mathbb{Z}$-graded) ideal of $\mathcal{U}^+(\chi)$ and $\mathcal{U}^-(\chi)$, respectively. Then the following statements are equivalent.*

(1) (*Triangular decomposition of $\mathcal{U}(\chi)/(\mathcal{I}^+ + \mathcal{I}^-)$.*) *The multiplication map* $\mathrm{m} : \mathcal{U}^+(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi) \to \mathcal{U}(\chi)$ *induces an isomorphism*

$$\mathcal{U}^+(\chi)/\mathcal{I}^+ \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi)/\mathcal{I}^- \to \mathcal{U}(\chi)/\big(\mathcal{I}^+ + \mathcal{I}^-\big)$$

of vector spaces.

(2) *The following equation holds.*

$$\mathcal{U}(\chi)\mathcal{I}^+\mathcal{U}(\chi) + \mathcal{U}(\chi)\mathcal{I}^-\mathcal{U}(\chi) = \mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) + \mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{I}^-.$$

(3) *The vector spaces $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ and $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{I}^-$ are ideals of $\mathcal{U}(\chi)$.*

(4) *For all $X \in \mathcal{U}^0(\chi)$ and $i \in I$ one has*

$$\begin{aligned} X \cdot \mathcal{I}^+ &\subset \mathcal{I}^+, & X \cdot \mathcal{I}^- &\subset \mathcal{I}^-, \\ \partial_i^K\big(\mathcal{I}^+\big) &\subset \mathcal{I}^+, & \partial_i^L\big(\mathcal{I}^+\big) &\subset \mathcal{I}^+, \\ \partial_i^K\big(\phi_4\big(\mathcal{I}^-\big)\big) &\subset \phi_4\big(\mathcal{I}^-\big), & \partial_i^L\big(\phi_4\big(\mathcal{I}^-\big)\big) &\subset \phi_4\big(\mathcal{I}^-\big). \end{aligned}$$

Proof. (1)$\Leftrightarrow$(2). The map in part (1) is surjective by the triangular decomposition of $\mathcal{U}(\chi)$. The injectivity of the map in part (1) means precisely that part (2) is true.

(3)$\Rightarrow$(2). This follows from the triangular decomposition of $\mathcal{U}(\chi)$.

(2)$\Rightarrow$(4). By the triangular decomposition of $\mathcal{U}(\chi)$, the linear map

$$\zeta^+ : \mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) \to \mathcal{U}^+(\chi)\mathcal{U}^0(\chi), \quad abc \mapsto ab\varepsilon(c),$$

where $a \in \mathcal{U}^+(\chi)$, $b \in \mathcal{U}^0(\chi)$, and $c \in \mathcal{U}^-(\chi)$, is a well-defined surjective linear map from $\mathcal{U}(\chi)$ to $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)$. The equation in part (2) and the standing assumption $\mathcal{I}^- \subset \ker \varepsilon$ imply that

$$\zeta^+\big(\mathcal{U}(\chi)\mathcal{I}^+\mathcal{U}(\chi) + \mathcal{U}(\chi)\mathcal{I}^-\mathcal{U}(\chi)\big) = \mathcal{I}^+\mathcal{U}^0(\chi).$$

Since $\zeta^+|_{\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)}$ is injective and $\mathcal{I}^+ \subset \mathcal{U}^+(\chi)$, the above equation implies that

$$\mathcal{U}^+(\chi)\mathcal{U}^0(\chi) \cap \big(\mathcal{U}(\chi)\mathcal{I}^+\mathcal{U}(\chi) + \mathcal{U}(\chi)\mathcal{I}^-\mathcal{U}(\chi)\big) = \mathcal{I}^+\mathcal{U}^0(\chi), \tag{4.41}$$

$$\mathcal{U}^+(\chi) \cap \big(\mathcal{U}(\chi)\mathcal{I}^+\mathcal{U}(\chi) + \mathcal{U}(\chi)\mathcal{I}^-\mathcal{U}(\chi)\big) = \mathcal{I}^+. \tag{4.42}$$

Now let $X \in \mathcal{U}^0(\chi)$ and $E \in \mathcal{I}^+$. Since $\mathcal{U}^0(\chi)$ is a group algebra, for the proof of the first two relations in part (4) one can assume that X is a group-like element. Then $XEX^{-1} \in \mathcal{U}^+(\chi)$ by Eqs. (4.21), and hence $XEX^{-1} \in \mathcal{I}^+$ by Eq. (4.42). Similarly one gets $X \cdot \mathcal{I}^- \subset \mathcal{I}^-$ for all $X \in \mathcal{U}^0(\chi)$.

Let again $E \in \mathcal{I}^+$. By Lemma 4.15 and Eq. (4.41) one has

$$\partial_i^K(E)K_i - L_i\partial_i^L(E) \in \mathcal{I}^+\mathcal{U}^0(\chi).$$

By triangular decomposition of $\mathcal{U}(\chi)$ and Eqs. (4.21) one obtains that $\partial_i^K(\mathcal{I}^+) \subset \mathcal{I}^+$ and $\partial_i^L(\mathcal{I}^+) \subset \mathcal{I}^+$. Finally, notice that the pair $(\mathcal{I}^+, \mathcal{I}^-)$ can be replaced by the pair $(\phi_4(\mathcal{I}^-), \phi_4(\mathcal{I}^+))$, and by definition of ϕ_4 the equation in part (2) holds for $(\mathcal{I}^+, \mathcal{I}^-)$ if and only if it holds for $(\phi_4(\mathcal{I}^-), \phi_4(\mathcal{I}^+))$. This symmetry yields immediately the remaining relations in part (4).

(4) $\Rightarrow$ (3). We prove first that $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is an ideal of $\mathcal{U}(\chi)$. Since $\mathcal{I}^+$ is a right ideal of $\mathcal{U}^+(\chi)$, triangular decomposition of $\mathcal{U}(\chi)$ implies that $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) = \mathcal{I}^+\mathcal{U}(\chi)$ is a right ideal of $\mathcal{U}(\chi)$. Since $\mathcal{I}^+$ is a left ideal of $\mathcal{U}^+(\chi)$, one obtains that

$$\mathcal{U}^+(\chi)\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) \subset \mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi).$$

Let $X \in \{K_i, L_i, F_i \mid i \in I\}$. The relation

$$X\mathcal{I}^+ \in \mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$$

follows immediately from Lemma 4.15 and the relations in part (4). Thus $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is also a left ideal of $\mathcal{U}(\chi)$. By the same arguments one gets that $\phi_4(\mathcal{I}^-)\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is an ideal of $\mathcal{U}(\chi)$. Apply the algebra antiautomorphism ϕ_4 to this fact to obtain that $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{I}^-$ is an ideal of $\mathcal{U}(\chi)$. □

Remark 4.18. Assume that $\mathcal{I} = (\mathcal{I}^+, \mathcal{I}^-)$ is an ideal of $\mathcal{U}(\chi)$ as in Proposition 4.17. Because of Proposition 4.17(2), see Eq. (4.42), the ideals $\mathcal{I}^+ \subset \mathcal{U}^+(\chi)$ and $\mathcal{I}^- \subset \mathcal{U}^-(\chi)$ are uniquely determined by $\mathcal{I}$. Explicitly, $\mathcal{I}^+ = \mathcal{U}^+(\chi) \cap \mathcal{I}$ and $\mathcal{I}^- = \mathcal{U}^-(\chi) \cap \mathcal{I}$.

Remark 4.19. Similarly to the proof of Proposition 4.17 one can show that the claim of Proposition 4.17(4) holds if and only if the multiplication map $\mathrm{m} : \mathcal{U}^-(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^+(\chi) \to \mathcal{U}(\chi)$ induces an isomorphism

$$\mathcal{U}^-(\chi)/\mathcal{I}^- \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^+(\chi)/\mathcal{I}^+ \to \mathcal{U}(\chi)/\big(\mathcal{I}^+ + \mathcal{I}^-\big)$$

of vector spaces.

Let $\pi_1 : \mathcal{U}^+(\chi) \to V^+(\chi) = \mathcal{U}^+(\chi)_1$ denote the surjective $\mathbb{Z}$-graded map, see Eqs. (4.25), with $\pi_1(E_i) = E_i$ for all $i \in I$. For all $j \in I$ let $E_j^* \in V^+(\chi)^*$ be the linear functional with $E_j^*(E_i) = \delta_{i,j}$ for all $i \in I$. Recall the braided Hopf algebra structure of $\mathcal{U}^+(\chi)$ given in Proposition 4.1.

Lemma 4.20. *For all $i \in I$ and $E \in \mathcal{U}^+(\chi)$*

$$\partial_i^K(E) = \big(\mathrm{id} \otimes E_i^* \circ \pi_1\big)\underline{\Delta}(E), \qquad \partial_i^L(E) = \big(E_i^* \circ \pi_1 \otimes \mathrm{id}\big)\underline{\Delta}(E),$$

where $\mathcal{U}^+(\chi) \otimes \Bbbk$ and $\Bbbk \otimes \mathcal{U}^+(\chi)$ are identified with $\mathcal{U}^+(\chi)$.

Proof. Both equations hold for $E \in \Bbbk \oplus V^+(\chi)$ by Eqs. (4.36). One checks easily that for the right-hand sides of the equations analogous formulas as Eqs. (4.37) hold. □

Corollary 4.21. *Let $\mathcal{I}^+ \subset \bigoplus_{m=2}^\infty \mathcal{U}^+(\chi)_m$ be a Yetter–Drinfel'd submodule (with respect to $\mathcal{U}^0(\chi)$, see Remark* 4.8*) and a biideal of $\mathcal{U}^+(\chi)$,* i.e. *$\mathcal{I}$ is an ideal and a braided coideal of $\mathcal{U}^+(\chi)$. Then $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is a Hopf ideal of $\mathcal{U}(\chi)$.*

Proof. The assumptions yield that $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is a coideal of $\mathcal{U}(\chi)$. Lemma 4.20 gives that $\partial_i^K(\mathcal{I}^+) \subset \mathcal{I}^+$ and $\partial_i^L(\mathcal{I}^+) \subset \mathcal{I}^+$ for all $i \in I$. Further, $X \cdot \mathcal{I}^+ \subset \mathcal{I}^+$ by assumption, and hence Proposition 4.17(4) $\Rightarrow$ (3) implies that $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is an ideal of $\mathcal{U}(\chi)$. Finally, $\mathcal{I}^+\mathcal{U}^0(\chi)$ is a Hopf ideal of $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)$ by a result of Takeuchi, see [Mon93, Lemma 5.2.10] and the corresponding remark in [AS02b, Section 2.1]. Thus $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is a Hopf ideal of $\mathcal{U}(\chi)$. □

4.3. Some relations of $\mathcal{U}(\chi)$

Let $\chi \in \mathcal{X}$ and let $p \in I$. For any $i \in I \setminus \{p\}$ let $E^+_{i,0(p)} = E^-_{i,0(p)} = E_i$, and for all $m \in \mathbb{N}$ define recursively

$$E^+_{i,m+1(p)} = E_p E^+_{i,m(p)} - \left(K_p \cdot E^+_{i,m(p)}\right) E_p, \tag{4.43}$$

$$E^-_{i,m+1(p)} = E_p E^-_{i,m(p)} - \left(L_p \cdot E^-_{i,m(p)}\right) E_p. \tag{4.44}$$

In connection with the variable p we will also write $E^+_{i,m}$ for $E^+_{i,m(p)}$ and $E^-_{i,m}$ for $E^-_{i,m(p)}$, where $m \in \mathbb{N}_0$. If somewhere p has to be replaced by another variable, then we will not use this abbreviation. Observe that $E^-_{i,m} = \phi_3\phi_2(E^+_{i,m})$, where $E^+_{i,m} \in \mathcal{U}((\chi^{-1})^{\mathrm{op}})$ and $E^-_{i,m} \in \mathcal{U}(\chi)$.

Using Eq. (3.2) and induction on m one can show that the explicit form of the elements $E^\pm_{i,m}$ is as follows.

$$E^+_{i,m} = \sum_{s=0}^{m} (-1)^s q_{pi}^s q_{pp}^{s(s-1)/2} \binom{m}{s}_{q_{pp}} E_p^{m-s} E_i E_p^s, \tag{4.45}$$

$$E^-_{i,m} = \sum_{s=0}^{m} (-1)^s q_{ip}^{-s} q_{pp}^{-s(s-1)/2} \binom{m}{s}_{q_{pp}^{-1}} E_p^{m-s} E_i E_p^s. \tag{4.46}$$

Lemma 4.22. *For all $i \in I \setminus \{p\}$ and all $m \in \mathbb{N}_0$*

$$\Bbbk E^+_{i,m+1} = \Bbbk\left(E^+_{i,m} E_p - \left(L_i L_p^m \cdot E_p\right) E^+_{i,m}\right),$$
$$\Bbbk E^-_{i,m+1} = \Bbbk\left(E^-_{i,m} E_p - \left(K_i K_p^m \cdot E_p\right) E^-_{i,m}\right).$$

Proof. The first equation of the lemma follows immediately from

$$L_i L_p^m \cdot E_p = q_{pi}^{-1} q_{pp}^{-m} E_p, \qquad K_p \cdot E^+_{i,m} = q_{pi} q_{pp}^m E^+_{i,m}.$$

To get the second equation, apply $\phi_3\phi_2$ to the first one. □

The following results are standard, see e.g. [Ros98,AS00,AS02a]. They are recalled to keep notational consistency and to have all variants needed for the sequel.

Lemma 4.23.

(i) *For all $m \in \mathbb{N}_0$*

$$\underline{\Delta}\left(E_p^m\right) = \sum_{r=0}^{m} \binom{m}{r}_{q_{pp}} E_p^r \otimes E_p^{m-r}.$$

(ii) *For all $i \in I \setminus \{p\}$ and all $m \in \mathbb{N}_0$*

$$\underline{\Delta}\left(E^+_{i,m}\right) = E^+_{i,m} \otimes 1 + \sum_{r=0}^{m} \binom{m}{r}_{q_{pp}} \prod_{s=1}^{r} \left(1 - q_{pp}^{m-s} q_{pi} q_{ip}\right) E_p^r \otimes E^+_{i,m-r},$$

$$\underline{\Delta}\left(E^-_{i,m}\right) = 1 \otimes E^-_{i,m} + \sum_{r=0}^{m} q_{pi}^r \binom{m}{r}_{q_{pp}} \prod_{s=1}^{r} \left(1 - q_{pp}^{s-m} q_{pi}^{-1} q_{ip}^{-1}\right) E^-_{i,m-r} \otimes E_p^r.$$

Proof. Use Proposition 4.1, Eq. (3.2), and induction on m. $\square$

Lemmata 4.23, 4.15 and 4.20 can be used to obtain commutation relations which will be essential to determine Lusztig isomorphisms between Drinfel'd doubles.

Corollary 4.24. *For all $m\in\mathbb{N}_0$ and $i,j\in I\setminus\{p\}$*

$$\begin{gathered}
\partial^K_p\big(E_p^m\big)=(m)_{q_{pp}}E_p^{m-1},\qquad \partial^K_i\big(E_p^m\big)=0,\\
\partial^L_p\big(E_p^m\big)=(m)_{q_{pp}}E_p^{m-1},\qquad \partial^L_i\big(E_p^m\big)=0,\\
\partial^K_j\big(E^+_{i,m}\big)=\delta_{i,j}\prod_{s=0}^{m-1}\big(1-q_{pp}^s q_{pi}q_{ip}\big)E_p^m,\qquad \partial^K_p\big(E^+_{i,m}\big)=0,\\
\partial^K_p\big(E^-_{i,m}\big)=(m)_{q_{pp}}\big(1-q_{pp}^{1-m}q_{pi}^{-1}q_{ip}^{-1}\big)E^-_{i,m-1},\qquad \partial^K_j\big(E^-_{i,m}\big)=\delta_{i,j}\delta_{m,0}1,\\
\partial^L_p\big(E^+_{i,m}\big)=(m)_{q_{pp}}\big(1-q_{pp}^{m-1}q_{pi}q_{ip}\big)E^+_{i,m-1},\qquad \partial^L_j\big(E^+_{i,m}\big)=\delta_{i,j}\delta_{m,0}1,\\
\partial^L_j\big(E^-_{i,m}\big)=\delta_{i,j}q_{pi}^m\prod_{s=0}^{m-1}\big(1-q_{pp}^{-s}q_{pi}^{-1}q_{ip}^{-1}\big)E_p^m,\qquad \partial^L_p\big(E^-_{i,m}\big)=0,
\end{gathered}$$

where products over s from 0 to -1 are interpreted as 1.

Corollary 4.25. *For all $m\in\mathbb{N}_0$ and all $i\in I\setminus\{p\}$*

$$\begin{aligned}
\big[E_p^m,F_p\big]&=(m)_{q_{pp}}\big(q_{pp}^{1-m}K_p-L_p\big)E_p^{m-1},\\
\big[E^+_{i,m},F_p\big]&=(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)L_pE^+_{i,m-1},\\
\big[E^+_{i,m},F_i\big]&=q_{ip}^{-m}\prod_{s=0}^{m-1}\big(1-q_{pp}^s q_{pi}q_{ip}\big)K_iE_p^m-\delta_{m,0}L_i,\\
\big[E^-_{i,m},F_p\big]&=q_{pp}^{1-m}(m)_{q_{pp}}\big(1-q_{pp}^{1-m}q_{pi}^{-1}q_{ip}^{-1}\big)K_pE^-_{i,m-1},\\
\big[E^-_{i,m},F_i\big]&=\delta_{m,0}K_i-q_{pi}^m\prod_{s=0}^{m-1}\big(1-q_{pp}^{-s}q_{pi}^{-1}q_{ip}^{-1}\big)L_iE_p^m,
\end{aligned}$$

where products over s from 0 to -1 are interpreted as 1. Moreover, $[E^+_{i,m},F_j]=[E^-_{i,m},F_j]=0$ for all $m\in\mathbb{N}_0$ and $i,j\in I\setminus\{p\}$ with $i\neq j$.

For $i\in I\setminus\{p\}$ and $m\in\mathbb{N}_0$ let

$$F^+_{i,m}=\phi_3\big(E^+_{i,m}\big),\qquad F^-_{i,m}=\phi_3\big(E^-_{i,m}\big),\tag{4.47}$$

where $E^+_{i,m}$, $E^-_{i,m}$ are elements of $\mathcal{U}^+(\chi^{\mathrm{op}})$. In particular,

$$\begin{aligned}
F^+_{i,0}&=F_i,\qquad F^+_{i,m+1}=F_pF^+_{i,m}-\big(L_p\cdot F^+_{i,m}\big)F_p,\\
F^-_{i,0}&=F_i,\qquad F^-_{i,m+1}=F_pF^-_{i,m}-\big(K_p\cdot F^-_{i,m}\big)F_p
\end{aligned}\tag{4.48}$$

for all $i\in I$ and $m\in\mathbb{N}_0$.

By induction on m one can show easily the following.

Lemma 4.26. *Let $i \in I \setminus \{p\}$. For all $\underline{a} \in (\Bbbk^\times)^I$, $n \in \mathbb{Z}$ and $m \in \mathbb{N}_0$*

$$\varphi_{\underline{a}}\big(E^\pm_{i,m}\big) \in \Bbbk^\times E^\pm_{i,m}, \qquad \varphi_n\big(E^\pm_{i,m}\big) \in \Bbbk^\times K_p^{mn} L_p^{-mn} K_i^n L_i^{-n} E^\pm_{i,m},$$
$$\varphi_{\underline{a}}\big(F^\pm_{i,m}\big) \in \Bbbk^\times F^\pm_{i,m}, \qquad \varphi_n\big(F^\pm_{i,m}\big) \in \Bbbk^\times K_p^{-mn} L_p^{mn} K_i^{-n} L_i^{n} F^\pm_{i,m}.$$

Further, for all $m \in \mathbb{N}_0$

$$\phi_1\big(E^\pm_{i,m}\big) \in \Bbbk^\times F^\pm_{i,m} L_i^{-1} L_p^{-m}, \qquad \phi_1\big(F^\pm_{i,m}\big) \in \Bbbk^\times K_i^{-1} K_p^{-m} E^\pm_{i,m},$$
$$\phi_2\big(E^\pm_{i,m}\big) = F^\mp_{i,m}, \qquad \phi_2\big(F^\pm_{i,m}\big) = (-1)^{m+1} E^\mp_{i,m},$$
$$\phi_3\big(E^\pm_{i,m}\big) = F^\pm_{i,m}, \qquad \phi_3\big(F^\pm_{i,m}\big) = E^\pm_{i,m},$$
$$\phi_4\big(E^\pm_{i,m}\big) \in \Bbbk^\times F^\mp_{i,m}, \qquad \phi_4\big(F^\pm_{i,m}\big) \in \Bbbk^\times E^\mp_{i,m}.$$

Lemma 4.27. *For all $i \in I \setminus \{p\}$ and all $m, n \in \mathbb{N}_0$ with $m \geqslant n$*

$$\begin{aligned}\big[E^+_{i,m}, F^+_{i,n}\big] = (-1)^n q_{ip}^{n-m} q_{pp}^{n(n-m)} \prod_{s=0}^{n-1} (m-s)_{q_{pp}} \prod_{s=0}^{m-1} \big(1 - q_{pp}^s q_{pi} q_{ip}\big) \\ \times \big(K_p^n K_i - \delta_{m,n} L_p^n L_i\big) E_p^{m-n}.\end{aligned}$$

Proof. Proceed by induction on n. The formula for $n = 0$ was proven in Corollary 4.25. Assume now that $m, n \in \mathbb{N}_0$ and $n < m$. Then

$$\begin{aligned}\big[E^+_{i,m}, F^+_{i,n+1}\big] &= \big[E^+_{i,m}, F_p F^+_{i,n} - q_{pp}^n q_{ip} F^+_{i,n} F_p\big] \\ &= \big[E^+_{i,m}, F_p\big] F^+_{i,n} + F_p \big[E^+_{i,m}, F^+_{i,n}\big] \\ &\quad - q_{pp}^n q_{ip} \big[E^+_{i,m}, F^+_{i,n}\big] F_p - q_{pp}^n q_{ip} F^+_{i,n} \big[E^+_{i,m}, F_p\big].\end{aligned} \tag{4.49}$$

Let

$$\alpha_{m,n} = (-1)^n q_{ip}^{n-m} q_{pp}^{n(n-m)} \prod_{s=0}^{n-1} (m-s)_{q_{pp}} \prod_{s=0}^{m-1} \big(1 - q_{pp}^s q_{pi} q_{ip}\big).$$

By induction hypothesis and Corollary 4.25 the sum of the second and third summands of the expression (4.49) is

$$\begin{aligned}&\alpha_{m,n}\big(F_p K_p^n K_i E_p^{m-n} - q_{pp}^n q_{ip} K_p^n K_i E_p^{m-n} F_p\big) \\ &\quad = -q_{pp}^n q_{ip} \alpha_{m,n} K_p^n K_i \big(E_p^{m-n} F_p - F_p E_p^{m-n}\big) \\ &\quad = -q_{pp}^n q_{ip} \alpha_{m,n} (m-n)_{q_{pp}} K_p^n K_i \big(q_{pp}^{1-m+n} K_p - L_p\big) E_p^{m-n-1} \\ &\quad = \alpha_{m,n+1} K_p^n K_i \big(K_p - q_{pp}^{m-n-1} L_p\big) E_p^{m-n-1}.\end{aligned}$$

Similarly, the sum of the first and fourth summands is equal to

$$
\begin{aligned}
&(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)\big(L_pE^+_{i,m-1}F^+_{i,n}-q_{pp}^nq_{ip}F^+_{i,n}L_pE^+_{i,m-1}\big)\\
&\quad=(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)L_p\big[E^+_{i,m-1},F^+_{i,n}\big]\\
&\quad=(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)\alpha_{m-1,n}L_p\big(K_p^nK_i-\delta_{m-1,n}L_p^nL_i\big)E_p^{m-1-n}\\
&\quad=q_{pp}^{m-n-1}\alpha_{m,n+1}L_p\big(K_p^nK_i-\delta_{m,n+1}L_p^nL_i\big)E_p^{m-1-n}.
\end{aligned}
$$

The latter two formulas imply the statement of the lemma for the expression $[E^+_{i,m},F^+_{i,n+1}]$. $\square$

Lemma 4.28. *Let $m,n\in\mathbb{N}_0$ and $i,j\in I\setminus\{p\}$ such that $i\neq j$. Then*

$$\big[E^+_{i,m},F^+_{j,n}\big]=0.$$

Proof. Proceed by induction on n. For $n=0$ the lemma follows from Corollary 4.25. Assume that $n\in\mathbb{N}_0$ with $[E^+_{i,m},F^+_{j,n}]=0$ for all $m\in\mathbb{N}_0$. Then

$$
\begin{aligned}
\big[E^+_{i,m},F^+_{j,n+1}\big]&=\big[E^+_{i,m},F_pF^+_{j,n}-q_{pp}^nq_{jp}F^+_{j,n}F_p\big]\\
&=\big[E^+_{i,m},F_p\big]F^+_{j,n}+F_p\big[E^+_{i,m},F^+_{j,n}\big]\\
&\quad-q_{pp}^nq_{jp}\big[E^+_{i,m},F^+_{j,n}\big]F_p-q_{pp}^nq_{jp}F^+_{j,n}\big[E^+_{i,m},F_p\big]\\
&=\big[E^+_{i,m},F_p\big]F^+_{j,n}-q_{pp}^nq_{jp}F^+_{j,n}\big[E^+_{i,m},F_p\big]\\
&=(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)\big(L_pE^+_{i,m-1}F^+_{j,n}-q_{pp}^nq_{jp}F^+_{j,n}L_pE^+_{i,m-1}\big)\\
&=(m)_{q_{pp}}\big(q_{pp}^{m-1}q_{pi}q_{ip}-1\big)L_p\big[E^+_{i,m-1},F^+_{j,n}\big]=0
\end{aligned}
$$

by induction hypothesis. $\square$

Definition 4.29. Let $p\in I$. Let $\mathcal{U}^+_{+p}(\chi)$ and $\mathcal{U}^+_{-p}(\chi)$ denote the subalgebra (with unit) of $\mathcal{U}^+(\chi)$ generated by $\{E^+_{j,m}\mid j\in I\setminus\{p\},\ m\in\mathbb{N}_0\}$ and $\{E^-_{j,m}\mid j\in I\setminus\{p\},\ m\in\mathbb{N}_0\}$, respectively.

Lemma 4.30. *Let $p\in I$.*

(i) *The algebras $\mathcal{U}^+_{+p}(\chi),\mathcal{U}^+_{-p}(\chi)$ are Yetter–Drinfel'd submodules of $\mathcal{U}^+(\chi)$ in ${}^{\mathcal{U}^0(\chi)}_{\mathcal{U}^0(\chi)}\mathcal{YD}$.*
(ii) *One has $\mathcal{U}^+_{+p}(\chi)\subset\ker\partial^K_p$ and $\mathcal{U}^+_{-p}(\chi)\subset\ker\partial^L_p$.*
(iii) *The algebra $\mathcal{U}^+_{+p}(\chi)$ is a left coideal of $\mathcal{U}^+(\chi)$ and the algebra $\mathcal{U}^+_{-p}(\chi)$ is a right coideal of $\mathcal{U}^+(\chi)$, that is*

$$
\begin{aligned}
\Delta\big(\mathcal{U}^+_{+p}(\chi)\big)&\subset\mathcal{U}^+(\chi)\otimes\mathcal{U}^+_{+p}(\chi),\\
\Delta\big(\mathcal{U}^+_{-p}(\chi)\big)&\subset\mathcal{U}^+_{-p}(\chi)\otimes\mathcal{U}^+(\chi).
\end{aligned}
$$

(iv) *Let $X\in\mathcal{U}^+_{+p}(\chi)$ and $Y\in\mathcal{U}^+_{-p}(\chi)$. Then*

$$E_pX-(K_p\cdot X)E_p\in\mathcal{U}^+_{+p}(\chi),\qquad E_pY-(L_p\cdot Y)E_p\in\mathcal{U}^+_{-p}(\chi).$$

Proof. Part (i) follows from the definition of $E^\pm_{i,m}$ and Eqs. (4.8) and (4.21). Part (ii) can be obtained from Eqs. (4.36), (4.37), part (i), and Corollary 4.24. Part (iii) follows immediately from Lemma 4.23(ii). Finally, consider the first relation of part (iv). First of all, this relation holds for all generators X

of $\mathcal{U}^+_{+p}(\chi)$ by definition of $E^+_{i,m}$. It is easy to see that if it holds for $X=X_1$ and $X=X_2$, then it also holds for $X=X_1X_2$. Thus the equation holds for all $X\in\mathcal{U}^+_{+p}(\chi)$. The second equation in part (iv) can be proven similarly. □

Lemma 4.31. *For all $p\in I$ the multiplication maps*

$$\mathrm{m}:\mathcal{U}^+_{+p}(\chi)\otimes\Bbbk[E_p]\to\mathcal{U}^+(\chi),\quad \mathrm{m}:\mathcal{U}^+_{-p}(\chi)\otimes\Bbbk[E_p]\to\mathcal{U}^+(\chi)$$

are isomorphisms of Yetter–Drinfel'd modules, where $\Bbbk[E_p]$ denotes the polynomial ring in one variable E_p.

Proof. We will prove surjectivity and injectivity of the first multiplication map. The proof for the second goes analogously.

The surjectivity of the first map follows from the facts that

- $E_i\in\mathcal{U}^+_{+p}(\chi)\Bbbk[E_p]$ for all $i\in I$,
- $\mathcal{U}^+_{+p}(\chi)\Bbbk[E_p]$ is a subalgebra of $\mathcal{U}(\chi)$ by Lemma 4.30(i), (iv).

Now we prove injectivity. Since $\mathcal{V}^+(\chi)$ is a $\mathbb{Z}^I$-graded Hopf algebra with $\mathcal{V}^+(\chi)_{m\alpha_p}=E_p^m\mathcal{U}^{+0}$ for all $m\in\mathbb{N}_0$, there is a unique $\mathbb{Z}^I$-graded retraction $\pi_{(p)}$ of the Hopf algebra embedding $\iota_{(p)}:\Bbbk[E_p]\#\mathcal{U}^{+0}\to\mathcal{V}^+(\chi)$. Thus $\mathcal{V}^+(\chi)$ is a right $\Bbbk[E_p]\#\mathcal{U}^{+0}$-Hopf module, see [Mon93, Definition 1.9.1], where the right module structure comes from multiplication and the right coaction is $(\mathrm{id}\otimes\pi_{(p)})\Delta$. Further, the elements of $\mathcal{U}^+_{+p}(\chi)$ are right coinvariant by Lemmata 4.23(ii), 4.30(i) and Remark 4.2. Thus m is injective by the fundamental theorem of Hopf modules [Mon93, 1.9.4]. □

Lemma 4.32. *Let $p\in I$ such that χ is p-finite. Let $i\in I\setminus\{p\}$ and $c_{pi}=c^\chi_{pi}$. Then*

$$E^+_{i,1-c_{pi}}-E^-_{i,1-c_{pi}}\in\Bbbk E_iE_p^{1-c_{pi}},\qquad F^+_{i,1-c_{pi}}-F^-_{i,1-c_{pi}}\in\Bbbk F_iF_p^{1-c_{pi}}.$$

If $(1-c_{pi})^!_{q_{pp}}\neq 0$, then both expressions are zero.

Proof. Lemma 4.31 and Eqs. (4.45), (4.46) imply that there exist $a_s\in\Bbbk$, where $1\leqslant s\leqslant 1-c_{pi}$, such that

$$E^-_{i,1-c_{pi}}=E^+_{i,1-c_{pi}}+\sum_{s=1}^{1-c_{pi}}a_sE^+_{i,1-c_{pi}-s}E^s_p.$$

Apply ∂^K_p to this expression. By Corollary 4.24 one gets $\partial^K_p(E^-_{i,1-c_{pi}})=0$ because of the definition of c_{pi}. By Lemmata 4.15 and 4.30(ii),

$$\partial^K_p\big(E^-_{i,1-c_{pi}}\big)=\sum_{s=1}^{1-c_{pi}}a_sE^+_{i,1-c_{pi}-s}\partial^K_p\big(E^s_p\big)=\sum_{s=1}^{1-c_{pi}}a_s(s)_{q_{pp}}E^+_{i,1-c_{pi}-s}E^{s-1}_p.$$

If $1\leqslant s\leqslant -c_{pi}$, then $(s)_{q_{pp}}\neq 0$ by definition of c_{pi}. Therefore Lemma 4.31 implies that $a_s=0$ whenever $1\leqslant s\leqslant -c_{pi}$. Further, if $(1-c_{pi})^!_{q_{pp}}\neq 0$, then also $a_{1-c_{pi}}=0$ by the same reason. This gives the statement of the lemma for $E^+_{i,1-c_{pi}}-E^-_{i,1-c_{pi}}$. The statement for $F^+_{i,1-c_{pi}}-F^-_{i,1-c_{pi}}$ follows from this by applying the isomorphism ϕ_3 and using Lemma 4.26. □

5. Nichols algebras of diagonal type

In this section some facts about Nichols algebras $\mathfrak{B}(V)$ of Yetter–Drinfel'd modules $V \in {}^H_H\mathcal{YD}$ are recalled, where H is a Hopf algebra. These (braided Hopf) algebras are named by W.D. Nichols who initiated the study of them [Nic78]. More details can be found e.g. in [AS02b, Section 2.1] and [Tak05]. Here it will be shown that the Drinfel'd double $\mathcal{U}(\chi)$ admits a natural quotient which is the Drinfel'd double of the Hopf algebras $\mathfrak{B}(V^+(\chi)) \# \mathcal{U}^{+0}$ and $\mathfrak{B}(V^-(\chi)) \# \mathcal{U}^{-0}$. These results generalize the corresponding statements in [Jos95, Section 3.1].

Definition 5.1. Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$ a finite-dimensional vector space over $\Bbbk$. The tensor algebra TV is a braided Hopf algebra in the Yetter–Drinfel'd category ${}^H_H\mathcal{YD}$, where the coproduct is defined by

$$\underline{\Delta}(v) = v \otimes 1 + 1 \otimes v \quad \text{for all } v \in V.$$

Let $\mathcal{S}$ be a maximal one among all braided coideals of TV contained in $\bigoplus_{n\geqslant 2} T^nV$, that is,

$$\underline{\Delta}(\mathcal{S}) \subset \mathcal{S} \otimes TV + TV \otimes \mathcal{S}.$$

(The assumption in [AS02b, Section 2.1] that $\mathcal{S}$ is a maximal braided biideal contained in $\bigoplus_{n\geqslant 2} T^nV$ leads to the same definition.) Then $\mathcal{S}$ is uniquely determined and it is a braided Hopf ideal of TV in the category ${}^H_H\mathcal{YD}$ (see also the arguments in the proof of Lemma 5.2). The quotient braided Hopf algebra $\mathfrak{B}(V) = TV/\mathcal{S}$ is termed the *Nichols algebra of* V. If H is the group algebra of an abelian group and V is semisimple, then $\mathfrak{B}(V)$ is called a *Nichols algebra of diagonal type*.

The following two statements have analogs for arbitrary Hopf algebras H and (finite-dimensional) Yetter–Drinfel'd modules $V \in {}^H_H\mathcal{YD}$. For convenience we will state the versions needed in this paper and also give short proofs. Note that $\mathbb{Z}^I$-grading is not discussed in the main reference [AS02b]. Moreover, the Nichols algebra with the definition there is $\mathbb{Z}$-homogeneous by definition, whereas here it follows from the maximality of $\mathcal{S}^+$.

Lemma 5.2. *Let $\chi \in \mathcal{X}$. The maximal coideal $\mathcal{S}^+(\chi)$ of $\mathcal{U}^+(\chi) \in {}^{\mathcal{U}^0(\chi)}_{\mathcal{U}^0(\chi)}\mathcal{YD}$ from Definition* 5.1 *is a Yetter–Drinfel'd submodule of $\mathcal{U}^+(\chi)$ and is a homogeneous ideal of $\mathcal{U}^+(\chi)$ with respect to the $\mathbb{Z}^I$-grading.*

Proof. Since the action and coaction of $\mathcal{U}^0(\chi)$ on $\mathcal{U}^+(\chi)$ are homogeneous with respect to the standard grading, the smallest Yetter–Drinfel'd submodule of $\mathcal{U}^+(\chi)$ containing $\mathcal{S}^+(\chi)$ is a coideal of $\mathcal{U}^+(\chi)$ consisting of elements of degree at least 2. Thus the maximality of $\mathcal{S}^+(\chi)$ implies that the coideal $\mathcal{S}^+(\chi)$ is a Yetter–Drinfel'd submodule of $\mathcal{U}^+(\chi)$.

The coproduct $\underline{\Delta}$ is a homogeneous map of degree 0. It is easy to see that for any coideal $\mathcal{I} \subset \bigoplus_{n=2}^\infty \mathcal{U}^+(\chi)_n$ the vector space $\bigoplus_{\mu\in\mathbb{Z}^I} \mathrm{pr}_\mu(\mathcal{I}) \supset \mathcal{I}$ is a coideal of $\mathcal{U}^+(\chi)$, where pr_μ is the homogeneous projection onto the homogeneous subspace of $\mathcal{U}^+(\chi)$ of degree $\mu \in \mathbb{Z}^I$. By the maximality assumption one obtains that $\mathcal{S}^+(\chi) = \bigoplus_{\mu\in\mathbb{Z}^I} \mathrm{pr}_\mu(\mathcal{S}^+(\chi))$. □

The Nichols algebra $\mathcal{U}^+(\chi)/\mathcal{S}^+(\chi)$ is denoted usually by $\mathfrak{B}(V^+(\chi))$. Later on, following the standard notation for quantized enveloping algebras, it will be more convenient to write $U^+(\chi)$ instead of $\mathfrak{B}(V^+(\chi))$. The coideal structure of $\mathcal{S}^+(\chi)$ induces a $U^+(\chi)$-bicomodule structure on $\mathcal{U}^+(\chi)$. The left and right coactions can be defined by

$$\delta_{\mathrm{l}}(X) = (\Pi \otimes \mathrm{id})\underline{\Delta}, \qquad \delta_{\mathrm{r}}(X) = (\mathrm{id} \otimes \Pi)\underline{\Delta}, \tag{5.1}$$

where $\Pi : \mathcal{U}^+(\chi) \to U^+(\chi)$ is the canonical surjection of braided Hopf algebras.

Proposition 5.3. *Let $X \in \mathcal{U}^+(\chi)$. The following are equivalent.*

(1) $X \in \mathcal{S}^+(\chi)$.
(2) $\varepsilon(X) = 0$ *and* $\partial_p^K(X) \in \mathcal{S}^+(\chi)$ *for all* $p \in I$.
(3) $\varepsilon(X) = 0$ *and* $(\Pi \otimes \pi_1)\underline{\Delta}(X) = 0$.
(4) $\varepsilon(X) = 0$ *and* $\partial_p^L(X) \in \mathcal{S}^+(\chi)$ *for all* $p \in I$.
(5) $\varepsilon(X) = 0$ *and* $(\pi_1 \otimes \Pi)\underline{\Delta}(X) = 0$.

Proof. Implications (1) ⇒ (2) and (1) ⇒ (4) follow from Lemma 4.20 and the assumption $\mathcal{S}^+(\chi) \subset \bigoplus_{n\geqslant 2} T^n V^+(\chi)$. Lemma 4.20 also yields the implications (2) ⇒ (3) and (4) ⇒ (5). We are content with giving a proof for the implication (3) ⇒ (1), the one for (5) ⇒ (1) being similar.

By Lemma 5.2 the coideal $\mathcal{S}^+(\chi)$ is $\mathbb{Z}$-homogeneous with respect to the standard grading of $\mathcal{U}^+(\chi)$. Further, $\underline{\Delta}$, Π and π_1 are $\mathbb{Z}$-homogeneous maps. Thus we may assume additionally that X is $\mathbb{Z}$-homogeneous. Since $\Pi(1) = 1$, (3) implies that the $\mathbb{Z}$-degree of X is at least 2. Let C be the left $U^+(\chi)$-subcomodule of $\mathcal{U}^+(\chi)$ generated by X. Then C is $\mathbb{Z}$-graded, since $\underline{\Delta}$ is $\mathbb{Z}$-homogeneous. One gets

$$\begin{aligned}\Pi\big(X^{(1)}\big) \otimes (\Pi \otimes \pi_1)\underline{\Delta}\big(X^{(2)}\big) &= \Pi\big(X^{(1)}\big) \otimes \Pi\big(X^{(2)}\big) \otimes \pi_1\big(X^{(3)}\big)\\ &= \underline{\Delta}\big(\Pi\big(X^{(1)}\big)\big) \otimes \pi_1\big(X^{(2)}\big) = 0.\end{aligned}$$

Let $C^+ = \{Y - \varepsilon(Y)1 \mid Y \in C\}$. Then $X \in C^+$, and by the above equation all elements of C^+ satisfy (3). Hence $C^+ \subset \bigoplus_{n\geqslant 2} \mathcal{U}^+(\chi)_n$. Further,

$$\underline{\Delta}\big(C^+\big) \subset \big(\mathcal{S}^+(\chi) + C^+\big) \otimes \mathcal{U}^+(\chi) + \mathcal{U}^+(\chi) \otimes C^+$$

and hence $\mathcal{S}^+(\chi) + C^+$ is a coideal of $\mathcal{U}^+(\chi)$. By maximality of $\mathcal{S}^+(\chi)$ one obtains that $X \in C^+ \subset \mathcal{S}^+(\chi)$ which proves statement (1). □

Proposition 5.3 yields a convenient characterization of the ideal $\mathcal{S}^+(\chi)$.

Proposition 5.4. *The following ideals of $\mathcal{U}^+(\chi)$ coincide.*

(1) *The ideal $\mathcal{S}^+(\chi)$.*
(2) *Any maximal element in the set of all ideals $\mathcal{I}^+$ of $\mathcal{U}^+(\chi)$ with*

$$\varepsilon\big(\mathcal{I}^+\big) = \{0\}, \qquad \partial_p^K\big(\mathcal{I}^+\big) \subset \mathcal{I}^+ \quad \textit{for all } p \in I.$$

(3) *Any maximal element in the set of all ideals $\mathcal{I}^+$ of $\mathcal{U}^+(\chi)$ with*

$$\varepsilon\big(\mathcal{I}^+\big) = \{0\}, \qquad \partial_p^L\big(\mathcal{I}^+\big) \subset \mathcal{I}^+ \quad \textit{for all } p \in I.$$

Proof. Proposition 5.3 implies that $\mathcal{S}^+(\chi)$ satisfies the properties of (2) and (3). It remains to show that any ideal in (2) respectively (3) coincides with $\mathcal{S}^+(\chi)$. We give an indirect proof for the ideals in (2). The ideals in (3) can be treated in a similar way.

Let $\mathcal{I}^+$ be maximal as in (2). Since ∂_p^K is homogeneous of degree -1 with respect to the standard grading of $\mathcal{U}^+(\chi)$, the vector space $\bigoplus_{n=0}^\infty \pi_n(\mathcal{I}^+)$ becomes an ideal of $\mathcal{U}^+(\chi)$ containing $\mathcal{I}^+$ and satisfying the conditions in (2). Thus the maximality of $\mathcal{I}^+$ implies that $\mathcal{I}^+$ is homogeneous with respect to the standard grading. Further, the assumptions in (2) imply that $\mathcal{I}^+ \subset \bigoplus_{n=2}^\infty \mathcal{U}^+(\chi)_n$. By a similar argument, using also Proposition 5.3(1)⇒(2), one obtains that $\mathcal{I}^+$ contains $\mathcal{S}^+(\chi)$. Assume now that $\mathcal{I}^+ \neq \mathcal{S}^+(\chi)$. Let $E \in \mathcal{I}^+$ be a homogeneous element of minimal degree, say n, with $E \notin \mathcal{S}^+(\chi)$.

Then $n \geqslant 2$, and $\partial_p^K(E) \in \mathcal{I}^+ \cap \mathcal{U}^+(\chi)_{n-1} = \mathcal{S}^+(\chi) \cap \mathcal{U}^+(\chi)_{n-1}$ for all $p \in I$. Hence $E \in \mathcal{S}^+(\chi)$ by Proposition 5.3(2) $\Rightarrow$ (1). This is a contradiction. □

Besides the properties in Lemma 5.2, the braided Hopf ideal $\mathcal{S}^+(\chi)$ has the following additional symmetries.

Lemma 5.5. *Let $\chi \in \mathcal{X}$. For all $m \in \mathbb{Z}$*

$$\varphi_m\big(\mathcal{S}^+(\chi)\big)\mathcal{U}^0(\chi) = \mathcal{S}^+(\chi)\mathcal{U}^0(\chi), \tag{5.2}$$

$$\phi_2\big(\mathcal{S}^+\big(\chi^{-1}\big)\big) = \phi_3\big(\mathcal{S}^+\big(\chi^{\mathrm{op}}\big)\big) = \phi_4\big(\mathcal{S}^+(\chi)\big), \tag{5.3}$$

$$\phi_1\big(\mathcal{S}^+(\chi)\big)\mathcal{U}^0(\chi) = \phi_4\big(\mathcal{S}^+(\chi)\big)\mathcal{U}^0(\chi). \tag{5.4}$$

Proof. Consider first Eq. (5.2). By definition of φ_m we obtain that $\varphi_m(\mathcal{S}^+(\chi))\mathcal{U}^0(\chi) = \varphi_m(\mathcal{S}^+(\chi)\mathcal{U}^0(\chi))$. Thus in view of Proposition 4.12(i) it suffices to prove the claim for $m = 1$. This claim follows from Lemmata 4.11 and 5.2.

Since $\mathcal{S}^+(\chi)$ is a braided Hopf ideal of $\mathcal{U}^+(\chi)$, $\mathcal{S}^+(\chi)\mathcal{U}^0(\chi)$ is a Hopf ideal of $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)$. Thus Eq. (5.4) follows from Proposition 4.12 and Corollary 4.10.

By Proposition 4.12 and Lemma 5.2 it remains to prove that

$$\phi_3\phi_4\big(\mathcal{S}^+(\chi)\big) \subset \mathcal{S}^+\big(\chi^{\mathrm{op}}\big), \qquad \phi_2\phi_4\big(\mathcal{S}^+(\chi)\big) \subset \mathcal{S}^+\big(\chi^{-1}\big). \tag{5.5}$$

We show the first formula in (5.5). The proof of the other one is similar.

The proof is based on Proposition 5.3. For brevity write $\phi = \phi_3\phi_4$. Consider the maps $\partial_p^K \circ \phi$ and $\phi \circ \partial_p^K$ as linear maps from $\mathcal{U}^+(\chi)$ to $\mathcal{U}^+(\chi^{\mathrm{op}})$. Lemma 4.15 and Proposition 4.9 imply that for all $p, i \in I$ and $X, Y \in \mathcal{U}^+(\chi)$

$$\begin{aligned}
\phi(K_p \cdot X) &= \phi\big(K_p X K_p^{-1}\big) = L_p^{-1}\phi(X)L_p = L_p^{-1} \cdot \phi(X),\\
\phi\big(\partial_p^K(E_i)\big) &= \partial_p^L\big(\phi(E_i)\big) = \delta_{p,i},\\
\phi\big(\partial_p^K(XY)\big) &= \big(L_p^{-1} \cdot \phi(Y)\big)\phi\big(\partial_p^K(X)\big) + \phi\big(\partial_p^K(Y)\big)\phi(X),\\
\partial_p^L\big(\phi(XY)\big) &= \partial_p^L\big(\phi(Y)\big)\phi(X) + \big(L_p^{-1} \cdot \phi(Y)\big)\partial_p^L\big(\phi(X)\big).
\end{aligned}$$

Hence for all $p \in I$

$$\phi_3\phi_4 \circ \partial_p^K = \partial_p^L \circ \phi_3\phi_4.$$

Thus the first formula in Eq. (5.5) holds by Proposition 5.3. □

Now the algebra $U(\chi)$ can be defined.

Proposition 5.6. *Let $\mathcal{S}^-(\chi) = \phi_4(\mathcal{S}^+(\chi))$. The vector space*

$$\mathcal{S}(\chi) = \mathcal{S}^+(\chi)\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) + \mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{S}^-(\chi)$$

is a Hopf ideal of $\mathcal{U}(\chi)$. The quotient Hopf algebra $\mathcal{U}(\chi)/\mathcal{S}(\chi)$ will be denoted by $U(\chi)$.

Proof. By Proposition 4.17(4) $\Rightarrow$ (2), Lemma 5.2, and Proposition 5.3, $\mathcal{S}(\chi)$ is an ideal of $\mathcal{U}(\chi)$. Using additionally Definition 5.1, $\mathcal{S}^+(\chi)\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ is a Hopf ideal of $\mathcal{U}(\chi)$. Similarly, $\mathcal{U}^0(\chi)\mathcal{S}^+(\chi^{\mathrm{op}})$ is a Hopf ideal of $\mathcal{U}^0(\chi)\mathcal{U}^+(\chi^{\mathrm{op}})$, and hence $\mathcal{U}^0(\chi)\mathcal{S}^-(\chi) = \phi_3(\mathcal{U}^0(\chi)\mathcal{S}^+(\chi^{\mathrm{op}}))$, see Lemma 5.5, is a Hopf ideal of $\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ by Proposition 4.9(6). Therefore $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{S}^-(\chi)$ is a Hopf ideal of $\mathcal{U}(\chi)$. $\square$

Remark 5.7. Suppose that $\chi \in \mathcal{X}$ is symmetric, *i.e.* $\chi = \chi^{\mathrm{op}}$. Then K_pL_p is for all $p \in I$ a central group-like element of the Hopf algebras $\mathcal{U}(\chi)$ and $U(\chi)$. In the example in Remark 4.7.1 the quantized symmetrizable Kac–Moody algebra is precisely $U(\chi)/(K_pL_p - 1 \mid p \in I)$, see also Theorem 5.8 below.

By Remark 4.18 one has $\mathcal{S}(\chi) \cap \mathcal{U}^+(\chi) = \mathcal{S}^+(\chi)$. Thus let

$$\begin{aligned}
U^+(\chi) &= \mathcal{U}^+(\chi) + \mathcal{S}(\chi)/\mathcal{S}(\chi) \cong \mathcal{U}^+(\chi)/\mathcal{S}^+(\chi),\\
U^-(\chi) &= \mathcal{U}^-(\chi) + \mathcal{S}(\chi)/\mathcal{S}(\chi) \cong \mathcal{U}^-(\chi)/\mathcal{S}^-(\chi),\\
U^+_{+p}(\chi) &= \mathcal{U}^+_{+p}(\chi) + \mathcal{S}(\chi)/\mathcal{S}(\chi) \cong \mathcal{U}^+_{+p}(\chi)/\big(\mathcal{S}^+(\chi) \cap \mathcal{U}^+_{+p}(\chi)\big),\\
U^+_{-p}(\chi) &= \mathcal{U}^+_{-p}(\chi) + \mathcal{S}(\chi)/\mathcal{S}(\chi) \cong \mathcal{U}^+_{-p}(\chi)/\big(\mathcal{S}^+(\chi) \cap \mathcal{U}^+_{-p}(\chi)\big).
\end{aligned}$$

Theorem 5.8. *The skew-Hopf pairing η in Proposition* 4.3 *induces a skew-Hopf pairing of the Hopf algebras $U^+(\chi) \# \mathcal{U}^{+0}$ and $(U^-(\chi) \# \mathcal{U}^{-0})^{\mathrm{cop}}$. The restriction of this pairing to $U^+(\chi) \times U^-(\chi)$ is non-degenerate.*

Proof. Recall that $\mathcal{S}^+(\chi)\mathcal{U}^{+0} \subset \bigoplus_{m=2}^{\infty} \mathcal{U}^+(\chi)_m\mathcal{U}^{+0}$ and $\mathcal{U}^{-0}\mathcal{S}^-(\chi) \subset \bigoplus_{m\leqslant -2} \mathcal{U}^{-0}\mathcal{U}^-(\chi)_m$, and that $\eta : \mathcal{U}^+(\chi)\mathcal{U}^{+0} \times \mathcal{U}^{-0}\mathcal{U}^-(\chi) \to \Bbbk$ is $\mathbb{Z}$-homogeneous. Hence

$$\begin{aligned}
\eta\big(\mathcal{S}^+(\chi)\mathcal{U}^{+0}, \mathcal{U}^{-0}\big(\Bbbk \oplus \mathcal{U}^-(\chi)_1\big)\big) &= 0,\\
\eta\big(\big(\Bbbk \oplus \mathcal{U}^+(\chi)_1\big)\mathcal{U}^{+0}, \mathcal{U}^{-0}\mathcal{S}^-(\chi)\big) &= 0.
\end{aligned}$$

By the arguments in the proof of Proposition 5.6, $\mathcal{S}^+(\chi)\mathcal{U}^{+0} \subset \mathcal{U}^+(\chi)\mathcal{U}^{+0}$ and $\mathcal{U}^{-0}\mathcal{S}^-(\chi) \subset \mathcal{U}^{-0}\mathcal{U}^-(\chi)$ are Hopf ideals. Then Eq. (4.12) gives that $\mathcal{S}^+(\chi)\mathcal{U}^{+0}$ is contained in the left radical and $\mathcal{U}^{-0}\mathcal{S}^-(\chi)$ is contained in the right radical of η. Thus η induces a skew-Hopf pairing of the Hopf algebras $U^+(\chi) \# \mathcal{U}^{+0}$ and $(U^-(\chi) \# \mathcal{U}^{-0})^{\mathrm{cop}}$. The left radical of the restriction of η to $\mathcal{U}^+(\chi) \times \mathcal{U}^-(\chi)$ is a braided coideal of the braided Hopf algebra $\mathcal{U}^+(\chi)$ by Eq. (4.12), Proposition 4.3(ii), and Remark 4.2. Moreover, it is $\mathbb{Z}$-homogeneous and contained in $\bigoplus_{m=2}^{\infty} \mathcal{U}^+(\chi)_m$ by Proposition 4.3(i). By definition, $\mathcal{S}^+(\chi)$ is the maximal such braided coideal of $\mathcal{U}^+(\chi)$, and hence $\mathcal{S}^+(\chi)$ is the left radical of the restriction of η to $\mathcal{U}^+(\chi) \times \mathcal{U}^-(\chi)$. Since $\dim \mathcal{U}^+(\chi)_m = \dim \mathcal{U}^-(\chi)_{-m} < \infty$ and $\dim \mathcal{S}^+(\chi)_m = \dim \phi_4(\mathcal{S}^+(\chi)_m) = \dim \mathcal{S}^-(\chi)_{-m}$ for all $m \in \mathbb{N}_0$, it follows that $\mathcal{S}^-(\chi)$ is the right radical of the restriction of η to $\mathcal{U}^+(\chi) \times \mathcal{U}^-(\chi)$. This proves the claim. $\square$

Corollary 5.9. *The Hopf algebra $U(\chi)$ is naturally isomorphic to the Drinfel'd double of the Hopf algebras $U^+(\chi) \# \mathcal{U}^{+0}$ and $(U^-(\chi) \# \mathcal{U}^{-0})^{\mathrm{cop}}$.*

By Proposition 5.3 the maps $\partial^K_p, \partial^L_p \in \mathrm{End}_\Bbbk(\mathcal{U}^+(\chi))$ induce $\Bbbk$-endomorphisms of $U^+(\chi)$ which will again be denoted by ∂^K_p and ∂^L_p, respectively. The following application of Lemma 4.30(ii) will be important in the next section.

Proposition 5.10. *For all $p \in I$ the following equations hold.*

$$\begin{aligned}
\ker\big(\partial^K_p : U^+(\chi) \to U^+(\chi)\big) &= U^+_{+p}(\chi),\\
\ker\big(\partial^L_p : U^+(\chi) \to U^+(\chi)\big) &= U^+_{-p}(\chi).
\end{aligned}$$

Proof. The inclusions "$\supset$" follow from Lemma 4.30(ii). By Lemma 4.31 and Eqs. (4.37) it suffices to show that $\partial_p^K(E_p^m)=0$ respectively $\partial_p^L(E_p^m)=0$ for some $m\in\mathbb{N}$ implies that $E_p^m=0$ in $U^+(\chi)$. By Corollary 4.24 one has $\partial_i^K(E_p^m)=\partial_i^L(E_p^m)=0$ for all $i\in I\setminus\{p\}$. Therefore Proposition 5.3 implies that for all $m\in\mathbb{N}$ the relations $E_p^m=0$, $\partial_p^K(E_p^m)=0$, and $\partial_p^L(E_p^m)=0$ are equivalent. □

6. Lusztig isomorphisms

One of our main goals in this paper is the construction of Lusztig isomorphisms between Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type, see Theorem 6.11. This is not possible for all $\chi\in\mathcal{X}$. Analogously to the quantized enveloping algebra setting, one has to assume that χ is p-finite for some $p\in I$, see Definition 3.11. Further, the proof of the existence of the Lusztig maps and their bijectivity is somewhat complex. Therefore first we introduce small ideals, with help of which Lusztig maps can be defined, see Lemma 6.6. This definition will then be used to induce isomorphisms between Drinfel'd doubles. In Section 6.3 many known relations for compositions of Lusztig automorphisms are generalized to our setting.

In the whole section let $\chi\in\mathcal{X}$, $q_{ij}=\chi(\alpha_i,\alpha_j)$, and $c_{ij}=c_{ij}^\chi$ for all $i,j\in I$.

6.1. Definition of Lusztig isomorphisms

Recall Eqs. (4.43)–(4.44) and the definition of h^χ from Eq. (3.16).

Definition 6.1. Let $p\in I$ and $h=h^\chi(\alpha_p)$. Assume that χ is p-finite. Let $\mathcal{I}_p^+(\chi)\subset\mathcal{U}^+(\chi)$ and $\mathcal{I}_p^-(\chi)\subset\mathcal{U}^-(\chi)$ be the following ideals. If $h^\chi(\alpha_p)<\infty$, then let

$$\mathcal{I}_p^+(\chi)=\big(E_p^h, E_{i,1-c_{pi}}^+\;\big|\;i\in I\setminus\{p\}\text{ such that }1-c_{pi}<h\big),$$
$$\mathcal{I}_p^-(\chi)=\big(F_p^h, F_{i,1-c_{pi}}^+\;\big|\;i\in I\setminus\{p\}\text{ such that }1-c_{pi}<h\big).$$

Otherwise define

$$\mathcal{I}_p^+(\chi)=\big(E_{i,1-c_{pi}}^+\;\big|\;i\in I\setminus\{p\}\big),\qquad \mathcal{I}_p^-(\chi)=\big(F_{i,1-c_{pi}}^+\;\big|\;i\in I\setminus\{p\}\big).$$

Proposition 6.2. *Let $p\in I$. Assume that χ is p-finite. Let $h=h^\chi(\alpha_p)$.*

(i) *If $h<\infty$, then the following ideals of $\mathcal{U}^+(\chi)$ coincide.*
- $\mathcal{I}_p^+(\chi)$,
- $(E_p^h, E_{i,1-c_{pi}}^-\mid i\in I\setminus\{p\}$ *such that* $1-c_{pi}<h)$,
- $(E_p^h, E_{i,1-c_{pi}}^+\mid i\in I\setminus\{p\})$,
- $(E_p^h, E_{i,1-c_{pi}}^-\mid i\in I\setminus\{p\})$.

(ii) *If $h=\infty$, then the following ideals of $\mathcal{U}^+(\chi)$ coincide.*
- $\mathcal{I}_p^+(\chi)$,
- $(E_{i,1-c_{pi}}^-\mid i\in I\setminus\{p\})$.

Proof. For both statements the equality of the first two ideals follows from Lemma 4.32. For the remaining assertions of part (i) of the lemma it suffices to show that if $1-c_{pi}\geqslant h$ (that is, $1-c_{pi}=h$ by definition of c_{pi}) then $E_{i,h}^+$ and $E_{i,h}^-$ are elements of $\mathcal{I}_p^+(\chi)$. The latter follows from the assumption $(h)_{q_{pp}}=0$, Lemma 3.2 and Eqs. (4.45), (4.46). □

The following lemma is a direct consequence of Lemma 4.26 and Proposition 6.2.

Lemma 6.3. *Let $p \in I$. Assume that χ is p-finite. Then the ideals $\mathcal{I}_p^{\pm}(\chi)$ are compatible with the automorphisms and antiautomorphism in Proposition* 4.9 *in the sense that*

$$\varphi_{\underline{a}}\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{I}_p^{\pm}(\chi),\qquad \phi_4\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{I}_p^{\mp}(\chi),$$
$$\phi_2\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{I}_p^{\mp}\big(\chi^{-1}\big),\qquad \phi_3\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{I}_p^{\mp}\big(\chi^{\mathrm{op}}\big),$$
$$\mathcal{U}^0(\chi)\varphi_m\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{U}^0(\chi)\mathcal{I}_p^{\pm}(\chi),\qquad \mathcal{U}^0(\chi)\phi_1\big(\mathcal{I}_p^{\pm}(\chi)\big)=\mathcal{U}^0(\chi)\mathcal{I}_p^{\mp}(\chi)$$

for all $\underline{a} \in (\Bbbk^{\times})^I$ and $m \in \mathbb{Z}$.

Further, Corollary 4.24 gives the following.

Lemma 6.4. *Let $p \in I$. Assume that χ is p-finite. Then for the ideals $\mathcal{I}_p^{\pm}(\chi)$ the equivalent statements in Proposition* 4.17 *hold.*

Lemma 6.5. *Let $p \in I$. Assume that χ is p-finite.*

(i) *The $\Bbbk$-endomorphism of $(\mathcal{U}^+_{+p}(\chi)+\mathcal{I}_p^+(\chi))/\mathcal{I}_p^+(\chi)$ given by $X \mapsto (\operatorname{ad} E_p)X$ is locally nilpotent.*
(ii) *The $\Bbbk$-endomorphism of $(\mathcal{U}^+_{-p}(\chi)+\mathcal{I}_p^+(\chi))/\mathcal{I}_p^+(\chi)$ given by $Y \mapsto E_pY-(L_p\cdot Y)E_p$ is locally nilpotent.*

Proof. The given maps are endomorphisms by the definition of $\mathcal{U}^+_{\pm p}(\chi)$. The statements of the lemma follow immediately from the following two facts. First, both $\Bbbk$-endomorphisms are in fact skew-derivations of the corresponding algebra $(\mathcal{U}^+_{\pm p}(\chi)+\mathcal{I}_p^+(\chi))/\mathcal{I}_p^+(\chi)$. Second, by the definitions of $E^{\pm}_{i,m}$ and $\mathcal{I}_p^+(\chi)$ and by Proposition 6.2 these skew-derivations are nilpotent on the corresponding algebra generators $E^{\pm}_{i,m}$. $\square$

Next we perform the first step towards the definition of Lusztig isomorphisms. Recall the definition of $\lambda_i(\chi)$ from Lemma 3.18.

Lemma 6.6. *Let $p \in I$. Assume that χ is p-finite. There are unique algebra maps*

$$T_p, T_p^- : \mathcal{U}(\chi) \to \mathcal{U}\big(r_p(\chi)\big)/\big(\mathcal{I}_p^+\big(r_p(\chi)\big), \mathcal{I}_p^-\big(r_p(\chi)\big)\big)$$

such that[2]

$$T_p(K_p)=T_p^-(K_p)=\underline{K}_p^{-1},\qquad T_p(K_i)=T_p^-(K_i)=\underline{K}_i\underline{K}_p^{-c_{pi}},$$
$$T_p(L_p)=T_p^-(L_p)=\underline{L}_p^{-1},\qquad T_p(L_i)=T_p^-(L_i)=\underline{L}_i\underline{L}_p^{-c_{pi}},$$
$$T_p(E_p)=\underline{F}_p\underline{L}_p^{-1},\qquad T_p(E_i)=\underline{E}^+_{i,-c_{pi}},$$
$$T_p(F_p)=\underline{K}_p^{-1}\underline{E}_p,\qquad T_p(F_i)=\lambda_i\big(r_p(\chi)\big)^{-1}\underline{F}^+_{i,-c_{pi}},$$
$$T_p^-(E_p)=\underline{K}_p^{-1}\underline{F}_p,\qquad T_p^-(E_i)=\lambda_i\big(r_p\big(\chi^{-1}\big)\big)^{-1}\underline{E}^-_{i,-c_{pi}},$$
$$T_p^-(F_p)=\underline{E}_p\underline{L}_p^{-1},\qquad T_p^-(F_i)=(-1)^{c_{pi}}\underline{F}^-_{i,-c_{pi}}.$$

[2] To avoid confusion in the proof of the lemma, the generators of $\mathcal{U}(r_p(\chi))$ are underlined. This convention will be used only in this subsection.

Proof. One has to show the compatibility of the definitions of T_p, T_p^- with the defining relations of $\mathcal{U}(\chi)$.

Let $\bar{q}_{ij} = r_p(\chi)(\alpha_i, \alpha_j)$ for all $i, j \in I$. The compatibility of T_p with the relations (4.19)–(4.22) is ensured (and enforced) by the choice of $r_p(\chi) \in \mathcal{X}$, see Eq. (3.13). The relation

$$\big[T_p(E_p), T_p(F_p)\big] = T_p(K_p - L_p)$$

is part of the proof of Proposition 4.9(4), since $T_p(E_p) = \phi_1(E_p)$, $T_p(F_p) = \phi_1(F_p)$, $T_p(K_p) = \phi_1(K_p)$, $T_p(L_p) = \phi_1(L_p)$. Further, for all $i \in I \setminus \{p\}$ one gets

$$\big[T_p(E_i), T_p(F_p)\big] = \big[\underline{E}^+_{i,-c_{pi}}, \underline{K}_p^{-1}\underline{E}_p\big] = -\underline{K}_p^{-1}\underline{E}^+_{i,1-c_{pi}} \in \mathcal{I}_p^+\big(r_p(\chi)\big)$$

because of Eqs. (4.34) and (4.43) and Proposition 6.2. Similarly,

$$\begin{aligned}\big[T_p(E_p), T_p(F_i)\big] &= \big[\underline{F}_p\underline{L}_p^{-1}, \lambda_i\big(r_p(\chi)\big)^{-1}\underline{F}^+_{i,-c_{pi}}\big] \\ &= \lambda_i\big(r_p(\chi)\big)^{-1}\bar{q}_{ip}^{-1}\bar{q}_{pp}^{c_{pi}}\underline{F}^+_{i,1-c_{pi}} \in \mathcal{I}_p^-\big(r_p(\chi)\big)\end{aligned}$$

by Eqs. (4.35) and (4.48).

Assume now that $i, j \in I \setminus \{p\}$ such that $i \neq j$. Then

$$\big[T_p(E_i), T_p(F_j)\big] = \big[\underline{E}^+_{i,-c_{pi}}, \lambda_j\big(r_p(\chi)\big)^{-1}\underline{F}^+_{j,-c_{pj}}\big] = 0 \tag{6.1}$$

by Lemma 4.28. On the other hand, for all $i \in I \setminus \{p\}$

$$\big[T_p(E_i), T_p(F_i)\big] = \big[\underline{E}^+_{i,-c_{pi}}, \lambda_i\big(r_p(\chi)\big)^{-1}\underline{F}^+_{i,-c_{pi}}\big] = T_p(K_i) - T_p(L_i) \tag{6.2}$$

by Lemma 4.27.

Similarly one can show that T_p^- is well defined. The relations

$$\big[\underline{E}^-_{i,-c_{pi}}, \underline{F}^-_{j,-c_{pj}}\big] = (-1)^{c_{pi}}\delta_{i,j}\lambda_i\big(r_p\big(\chi^{-1}\big)\big)T_p^-(K_i - L_i),$$

where $i, j \in I \setminus \{p\}$, follow from Eqs. (6.1) and (6.2) by applying the isomorphism ϕ_2 and using Lemma 4.26. $\square$

Let $p \in I$. Assume that χ is p-finite. In the next lemma and its proof we use the following abbreviations:

$$\mathcal{U}^+\big(r_p(\chi)\big)' = \big(\mathcal{U}^+\big(r_p(\chi)\big) + \mathcal{I}_p^+\big(r_p(\chi)\big)\big)/\mathcal{I}_p^+\big(r_p(\chi)\big), \tag{6.3}$$

$$\mathcal{U}^+_{\epsilon p}\big(r_p(\chi)\big)' = \big(\mathcal{U}^+_{\epsilon p}\big(r_p(\chi)\big) + \mathcal{I}_p^+\big(r_p(\chi)\big)\big)/\mathcal{I}_p^+\big(r_p(\chi)\big), \tag{6.4}$$

where $\epsilon \in \{+, -\}$. By Corollary 4.24 the skew-derivations ∂_p^K and ∂_p^L of $\mathcal{U}^+(r_p(\chi))$ induce well-defined skew-derivations on $\mathcal{U}^+(r_p(\chi))'$ which then will be denoted by the same symbol.

Lemma 6.7. *Let $p \in I$, T_p and T_p^- as in Lemma* 6.6. *Let $\bar{q}_{ij} = r_p(\chi)(\alpha_i, \alpha_j)$ for all $i, j \in I$.*

(a) *For all $X \in \mathcal{U}^+_{-p}(\chi)$ and $Y \in \mathcal{U}^+_{+p}(\chi)$*

$$T_p\big(E_p X - (L_p \cdot X)E_p\big) = \bar{q}_{pp}\partial^L_p\big(T_p(X)\big),$$
$$T_p^-\big(E_p Y - (K_p \cdot Y)E_p\big) = -\underline{K}_p^{-1} \cdot \partial^K_p\big(T_p^-(Y)\big).$$

(b) *For all $i \in I \setminus \{p\}$ and $t \in \mathbb{N}_0$ with $t \leqslant -c_{pi}$*

$$T_p\big(E^-_{i,t}\big) = \bar{q}^t_{pp}\prod_{s=0}^{t-1}(-c_{pi} - s)_{\bar{q}_{pp}}\prod_{s=1}^{t}\big(1 - \bar{q}_{pp}^{-c_{pi}-s}\bar{q}_{pi}\bar{q}_{ip}\big)\underline{E}^+_{i,-c_{pi}-t},$$
$$T_p^-\big(E^+_{i,t}\big) = \prod_{s=1}^{-c_{pi}-t}(s)^{-1}_{\bar{q}_{pp}^{-1}}\prod_{s=0}^{-c_{pi}-t-1}\big(\bar{q}_{pp}^{-s}\bar{q}_{pi}^{-1}\bar{q}_{ip}^{-1} - 1\big)^{-1}\underline{E}^-_{i,-c_{pi}-t}.$$

(c) *For all $i \in I \setminus \{p\}$ and $t \in \mathbb{N}_0$ with $t > -c_{pi}$*

$$T_p\big(E^-_{i,t}\big) = T_p^-\big(E^+_{i,t}\big) = 0.$$

(d) *The following relations hold.*

$$T_p\big(\mathcal{U}^+_{-p}(\chi)\big) \subset \mathcal{U}^+_{+p}\big(r_p(\chi)\big)', \qquad T_p^-\big(\mathcal{U}^+_{+p}(\chi)\big) \subset \mathcal{U}^+_{-p}\big(r_p(\chi)\big)'.$$

Proof. We start with a technical statement.

Step 1. *Part* (a) *holds for all $X, Y \in \mathcal{U}(\chi)$ with $T_p(X) \in \mathcal{U}^+_{+p}(r_p(\chi))'$ and $T_p^-(Y) \in \mathcal{U}^+_{-p}(r_p(\chi))'$.* Let $X \in \mathcal{U}(\chi)$. Assume that $T_p(X) \in \mathcal{U}^+_{+p}(r_p(\chi))'$. By the remark above the lemma, the expression $\partial^L_p(T_p(X))$ is well defined. By definition of T_p one gets

$$\begin{aligned}
T_p\big(E_p X - (L_p \cdot X)E_p\big) &= \underline{F}_p\underline{L}_p^{-1}T_p(X) - \big(\underline{L}_p^{-1} \cdot T_p(X)\big)\underline{F}_p\underline{L}_p^{-1}\\
&= -\big[\underline{L}_p^{-1} \cdot T_p(X), \underline{F}_p\big]\underline{L}_p^{-1}\\
&= \big(-\partial^K_p\big(\underline{L}_p^{-1} \cdot T_p(X)\big)\underline{K}_p + \underline{L}_p\partial^L_p\big(\underline{L}_p^{-1} \cdot T_p(X)\big)\big)\underline{L}_p^{-1}\\
&= \bar{q}_{pp}\partial^L_p\big(T_p(X)\big) - \bar{q}_{pp}\big(\underline{L}_p^{-1} \cdot \partial^K_p\big(T_p(X)\big)\big)\underline{K}_p\underline{L}_p^{-1}\\
&= \bar{q}_{pp}\partial^L_p\big(T_p(X)\big),
\end{aligned}$$

where the penultimate equation follows from Lemma 4.16 and the last one from the assumption $T_p(X) \in \mathcal{U}^+_{+p}(r_p(\chi))'$ and Lemma 4.30(ii). This and a similar calculation for $T_p^-(E_pY - (K_p \cdot Y)E_p)$ imply the statement of step 1.

Step 2. *Proof of parts* (b) *and* (c). We proceed by induction on t. For $t = 0$, part (b) is valid by the definitions of T_p and T_p^-. Assume now that the formulas in part (b) are valid for some $t < -c_{pi}$, where $i \in I \setminus \{p\}$. In view of Eqs. (4.43), (4.44) one can apply step 1 of the proof to $X = E^-_{i,t}$ and $Y = E^+_{i,t}$. Then one obtains part (b) for $T_p(E^-_{i,t+1})$ and $T_p^-(E^+_{i,t+1})$ from the induction hypothesis and Corollary 4.24. Similarly, if $t = -c_{pi}$, then the analogous induction step shows that $T_p(E^-_{i,-c_{pi}+1})$ is a multiple of $\partial^L_p(E_i) = 0$ and hence it is zero. This and a similar argument for $T_p^-(E^+_{i,-c_{pi}+1})$ imply part (c).

Step 3. *Proof of parts* (a) *and* (d). Since T_p and T_p^- are algebra maps, part (d) follows immediately from the definition of $\mathcal{U}^+_{\pm p}(\chi)$ and parts (b) and (c) of the lemma. Finally, part (a) is a direct consequence of step 1 of the proof and part (d) of the lemma. □

Proposition 6.8. *Let p, T_p and T_p^- as in Lemma* 6.6.

(i) *The maps T_p, T_p^- induce algebra isomorphisms*

$$T_p, T_p^- : \mathcal{U}(\chi)/\big(\mathcal{I}_p^+(\chi), \mathcal{I}_p^-(\chi)\big) \to \mathcal{U}\big(r_p(\chi)\big)/\big(\mathcal{I}_p^+\big(r_p(\chi)\big), \mathcal{I}_p^-\big(r_p(\chi)\big)\big).$$

(ii) *The isomorphisms T_p, T_p^- in* (i) *satisfy the equations*

$$\begin{gathered}
T_pT_p^- = T_p^-T_p = \mathrm{id},\\
T_p\varphi_{\underline{a}} = \varphi_{\underline{b}}T_p, \quad \textit{where } \underline{a} \in \big(\Bbbk^\times\big)^I, \qquad b_i = a_ia_p^{-c_{pi}} \quad \textit{for all } i \in I,\\
T_p^-\varphi_{\underline{a}} = \varphi_{\underline{b}}T_p^-, \quad \textit{where } \underline{a} \in \big(\Bbbk^\times\big)^I, \qquad b_i = a_ia_p^{-c_{pi}} \quad \textit{for all } i \in I,\\
T_p\phi_2 = \phi_2T_p^-\varphi_{\underline{a}}, \quad \textit{where } a_i = (-1)^{\delta_{i,p}} \textit{ for all } i \in I,\\
T_p\phi_3 = \phi_3T_p\varphi_{\underline{\lambda}}, \quad \textit{where } \lambda_p = q_{pp}^{-1},\ \lambda_i = \lambda_i\big(r_p(\chi)\big)^{-1} \textit{ for all } i \in I \setminus \{p\},\\
T_p^-\phi_3 = \phi_3T_p^-\varphi_{\underline{\lambda}}, \quad \textit{where } \lambda_p = q_{pp}^{-1},\ \lambda_i = (-1)^{c_{pi}}\lambda_i\big(r_p\big(\chi^{-1}\big)\big) \textit{ for all } i \in I \setminus \{p\},\\
T_p\phi_4 = \phi_4T_p^-\varphi_{\underline{a}} \quad \textit{for some } \underline{a} \in \big(\Bbbk^\times\big)^I.
\end{gathered}$$

Note that part (ii) makes only sense if one uses appropriate bicharacters. For example, the equation $T_pT_p^- = \mathrm{id}$ means that if T_p^- is defined with respect to χ, then T_p has to be defined with respect to $r_p(\chi)$. Similar adaptation has to be performed for the commutation relations with ϕ_2 and ϕ_3.

Proof of Proposition 6.8. First check that equation

$$T_p\phi_2(X) = \phi_2T_p^-\varphi_{\underline{a}}(X), \quad \text{where } a_i = (-1)^{\delta_{i,p}} \text{ for all } i \in I, \tag{6.5}$$

holds for all generators X of $\mathcal{U}(\chi)$. Since T_p, T_p^-, ϕ_2, and $\varphi_{\underline{a}}$ are algebra maps, this implies that

$$T_p\phi_2 = \phi_2T_p^-\varphi_{\underline{a}}, \quad \text{where } a_i = (-1)^{\delta_{i,p}} \text{ for all } i \in I. \tag{6.6}$$

Further, the equations $T_p\varphi_{\underline{a}} = \varphi_{\underline{b}}T_p$, $T_p^-\varphi_{\underline{a}} = \varphi_{\underline{b}}T_p^-$ as algebra maps $\mathcal{U}(\chi) \to \mathcal{U}(r_p(\chi))/(\mathcal{I}_p^+(r_p(\chi)), \mathcal{I}_p^-(r_p(\chi)))$ follow immediately from the definitions of T_p, T_p^- and $\varphi_{\underline{a}}$. Using Eq. (6.6) one can easily see with help of Lemmata 6.6, 6.7 and 4.26 that T_p and T_p^- are well defined on the given quotient of $\mathcal{U}(\chi)$. Again using Lemma 6.7 one gets that $T_pT_p^- = T_p^-T_p = \mathrm{id}$ and $T_p\phi_3 = \phi_3T_p\varphi_{\underline{\lambda}}$. The equation $T_p^-\phi_3 = \phi_3T_p^-\varphi_{\underline{\lambda}}$ follows from equations $T_p\phi_3 = \phi_3T_p\varphi_{\underline{\lambda}}$ and $T_p\phi_2 = \phi_2T_p^-\varphi_{\underline{a}}$ by Proposition 4.12. Equation $T_p\phi_4 = \phi_4T_p^-\varphi_{\underline{a}}$ can be obtained similarly to Eq. (6.6). □

6.2. *Lusztig isomorphisms for $U(\chi)$*

We continue to use the notation from Sections 5 and 6 and from Proposition 5.6.

Lemma 6.9. *Let $p \in I$. Assume that χ is p-finite.*

(i) *One has $\mathcal{I}_p^+(\chi) \subset \mathcal{S}^+(\chi)$.*
(ii) *Let $\epsilon \in \{+,-\}$. The ideal $\mathcal{S}^+(\chi)$ of $\mathcal{U}^+(\chi)$ is generated by the subset*

$$\big(\mathcal{S}^+(\chi) \cap \Bbbk[E_p]\big) \cup \big(\mathcal{S}^+(\chi) \cap \mathcal{U}^+_{\epsilon p}(\chi)\big).$$

Proof. (i) The generators of $\mathcal{I}_p^+(\chi)$ are in $\mathcal{S}^+(\chi)$ because of Corollary 4.24 and Proposition 5.3(2) $\Rightarrow$ (1). This implies the claim.

(ii) We consider the case $\epsilon = 1$, the proof for the other one is similar. Let $X \in \mathcal{S}^+(\chi)$. By Lemma 4.31 there exist $m \in \mathbb{N}_0$ and uniquely determined elements $X_0, \dots, X_m \in \mathcal{U}^+_{+p}(\chi)$ such that $X = \sum_{i=0}^m X_i E_p^i$. By Lemma 5.2 it suffices to consider the case when X is homogeneous with respect to the standard grading. Further, since $(n)^!_{q_{pp}} = 0$ implies that $E_p^n \in \mathcal{S}^+(\chi)$, see Proposition 5.3, one can assume that $(m)^!_{q_{pp}} \neq 0$, and that either $X = 0$ or $X_m \notin \mathcal{S}^+(\chi)$. Recall from Lemma 4.30(ii) that $\partial_p^K(X_i) = 0$ for all i. Therefore by Lemma 4.15 and Corollary 4.24 one gets

$$\sum_{i=0}^m \big(\partial_p^K\big)^m \big(X_i E_p^i\big) = (m)^!_{q_{pp}} X_m.$$

Thus Proposition 5.3 gives that $X_m \in \mathcal{S}^+(\chi)$. Hence $X = 0$, and (ii) is proven. $\square$

Proposition 6.10. *Let p, T_p and T_p^- as in Lemma* 6.6.

(i) *For all $i \in I \setminus \{p\}$ there exists $\underline{a} \in (\Bbbk^\times)^I$ such that*

$$\partial_i^L T_p = T_p \circ \big(\partial_p^L\big)^{-c_{pi}} \partial_i^L \varphi_{\underline{a}} \tag{6.7}$$

as a linear map $\mathcal{U}^+_{-p}(\chi) \to \mathcal{U}^+_{+p}(r_p(\chi))'$, see Eq. (6.4).
(ii) *For all $i \in I \setminus \{p\}$ there exists $\underline{a} \in (\Bbbk^\times)^I$ such that*

$$\partial_i^K T_p^- = T_p^- \circ \big(\partial_p^K\big)^{-c_{pi}} \partial_i^K \varphi_{\underline{a}}$$

as a linear map $\mathcal{U}^+_{+p}(\chi) \to \mathcal{U}^+_{-p}(r_p(\chi))'$.

Proof. We prove part (i) in 3 steps and leave the similar proof of part (ii) to the reader.

Step 1. *Eq.* (6.7) *holds on the generators of the algebra $\mathcal{U}^+_{-p}(\chi)$.* Let $j \in I \setminus \{p\}$ and $m \in \mathbb{N}_0$. If $m > -c_{pj}$, then for all $\underline{a} \in (\Bbbk^\times)^I$ the evaluations of both sides of Eq. (6.7) on $E^-_{j,m}$ give 0 by the following reasons. The left-hand side is zero by Lemma 6.7(c). For the right-hand side we obtain that $\varphi_{\underline{a}}(E^-_{j,m}) \in \Bbbk^\times E^-_{j,m}$ by Lemma 4.26. By the last line of Corollary 4.24 and the definition of c_{pj} in Definition 3.11 the expression $\partial_i^L(E^-_{j,m})$ can be non-zero only if $i = j$ and $(1 - c_{pi})_{q_{pp}} = 0$. In this case $\partial_i^L(E^-_{j,m}) \in \Bbbk E_p^m$. If $-c_{pi} < m < 1 - 2c_{pi}$ then $(\partial_p^L)^{-c_{pi}} \partial_i^L(E^-_{i,m}) = 0$ by the second line of Corollary 4.24. Otherwise $(\partial_p^L)^{-c_{pi}} \partial_i^L(E^-_{i,m}) \in \Bbbk E_p^{m+c_{pi}} \subset \mathcal{I}_p^+(\chi)$, and hence $T_p \circ (\partial_p^L)^{-c_{pi}} \partial_i^L(E^-_{i,m}) = 0$ by Proposition 6.8(i).

Assume now that $m \leqslant -c_{pj}$. If $j \neq i$, then Lemma 6.7(b) and Corollary 4.24 imply that $\partial_i^L T_p(E^-_{j,m}) = 0$, and the right-hand side of Eq. (6.7) vanishes on $E^-_{j,m}$ by the arguments in the previous paragraph. Suppose now that $j = i$. Then

$$\begin{aligned}\partial_i^L T_p\big(E_{i,m}^-\big) &\in \Bbbk^\times \partial_i^L\big(E_{i,-c_{pi}-m}^+\big) = \Bbbk^\times \delta_{m,-c_{pi}},\\ T_p\big(\big(\partial_p^L\big)^{-c_{pi}}\partial_i^L\varphi_{\underline{a}}\big(E_{i,m}^-\big)\big) &\in \Bbbk^\times T_p\big(\big(\partial_p^L\big)^{-c_{pi}}\partial_i^L\big(E_{i,m}^-\big)\big)\\ &= \Bbbk^\times T_p\big(\big(\partial_p^L\big)^{-c_{pi}}\big(E_p^m\big)\big)\\ &= \Bbbk^\times T_p(\delta_{m,-c_{pi}}) = \Bbbk^\times \delta_{m,-c_{pi}}.\end{aligned}$$

Step 2. *The map* $\vartheta_i = (\partial_p^L)^{-c_{pi}}\partial_i^L\varphi_{\underline{a}} \in \operatorname{End}_\Bbbk(\mathcal{U}_{-p}^+(\chi))$ *satisfies*

$$\vartheta_i\big(EE'\big) = \vartheta_i(E)E' + \big(L_p^{c_{pi}}L_i^{-1}\cdot E\big)\vartheta_i\big(E'\big) \quad \textit{for all } E, E' \in \mathcal{U}_{-p}^+(\chi).$$

The statement follows immediately from Eqs. (4.37), (4.39) and from $\partial_p^L(E) = \partial_p^L(E') = 0$, see Corollary 4.24.

Step 3. *Eq.* (6.7) *holds on* $\mathcal{U}_{-p}^+(\chi)$. In view of step 1 it suffices to show that if Eq. (6.7) holds on $E, E' \in \mathcal{U}_{-p}^+(\chi)$, then it also holds on EE'. Since T_p is an algebra map, the latter follows from Eq. (4.37), step 2 and equation $T_p(L_p^{c_{pi}}L_i^{-1}) = L_i^{-1}$. □

Theorem 6.11. *Let* p, T_p *and* T_p^- *as in Lemma* 6.6. *The maps* T_p, T_p^- *induce algebra isomorphisms*

$$T_p, T_p^- : U(\chi) \to U\big(r_p(\chi)\big).$$

The analogs of the commutation relations in Proposition 6.8 *hold.*

Proof. Extend the notation in Eqs. (6.3) and (6.4) by defining

$$\mathcal{S}^+(\chi)' = \mathcal{S}^+(\chi)/\mathcal{I}_p^+(\chi), \qquad \mathcal{S}(\chi)' = \mathcal{S}(\chi)/\big(\mathcal{I}_p^+(\chi), \mathcal{I}_p^-(\chi)\big).$$

In view of the commutation relations between T_p and ϕ_3 respectively T_p^- and ϕ_3 it suffices to show that $T_p(\mathcal{S}^+(\chi)) \subset \mathcal{S}(r_p(\chi))'$ and $T_p^-(\mathcal{S}^+(\chi)) \subset \mathcal{S}(r_p(\chi))'$. We prove the above relation for T_p. The proof for T_p^- goes similarly. Further, by Lemma 6.9 it suffices to show that

$$T_p\big(\mathcal{S}^+(\chi) \cap \mathcal{U}_{-p}^+(\chi)\big) \subset \mathcal{S}^+\big(r_p(\chi)\big)', \qquad T_p\big(\mathcal{S}^+(\chi) \cap \Bbbk[E_p]\big) \subset \mathcal{S}\big(r_p(\chi)\big)',$$

where the latter relation is obviously true since

$$\mathcal{S}^+(\chi) \cap \Bbbk[E_p] = \begin{cases} 0 & \text{if } h^\chi(\alpha_p) = \infty,\\ \sum_{m=h}^\infty \Bbbk E_p^m & \text{if } h = h^\chi(\alpha_p) < \infty.\end{cases} \tag{6.8}$$

Since $\mathcal{S}^+(\chi) \cap \mathcal{U}_{-p}^+(\chi)$ is $\mathbb{Z}^I$-graded, it is sufficient to show that $T_p(X) \in \mathcal{S}^+(r_p(\chi))'$ for any homogeneous element $X \in \mathcal{S}^+(\chi) \cap \mathcal{U}_{-p}^+(\chi)$. This can be done by induction on $|\mu|$, where $T_p(X) \in \mathcal{U}_{+p}^+(r_p(\chi))'_\mu$, see Lemma 6.7(d). The induction hypothesis is fulfilled since $T_p(X) \in \mathcal{U}_{+p}^+(r_p(\chi))'_0$ implies $X \in \Bbbk 1$, and hence $X \in \mathcal{S}^+(\chi)$ if and only if $X = 0$.

Let now $n \in \mathbb{N}$ and assume that relations $X \in \mathcal{S}^+(\chi) \cap \mathcal{U}_{-p}^+(\chi)$ and $T_p(X) \in \mathcal{U}_{+p}^+(r_p(\chi))'_\mu$ with $|\mu| \leqslant n$ imply that $T_p(X) \in \mathcal{S}^+(r_p(\chi))'$. Let $Y \in \mathcal{S}^+(\chi) \cap \mathcal{U}_{-p}^+(\chi)$ such that $T_p(Y) \in \mathcal{U}_{+p}^+(r_p(\chi))'_\mu$, where $|\mu| = n+1$. We have to show that $T_p(Y) \in \mathcal{S}^+(r_p(\chi))'_\mu$. By Proposition 5.3 this is equivalent to the relations $\partial_i^L(T_p(Y)) \in \mathcal{S}^+(r_p(\chi))'_{\mu-\alpha_i}$ for all $i \in I$. If $i = p$, then one gets from Lemma 6.7 that

$$\partial_p^L\big(T_p(Y)\big) = q_{pp}^{-1}T_p\big(E_pY - (L_p\cdot Y)E_p\big).$$

Since $E_pY-(L_p\cdot Y)E_p\in\mathcal{S}^+(\chi)\cap\mathcal{U}^+_{-p}(\chi)$, induction hypothesis implies that $\partial^L_p(T_p(Y))\in\mathcal{S}^+(r_p(\chi))'$. On the other hand, if $i\neq p$, then analogously Propositions 5.3 and 6.10, see also step 2 of the proof of the latter, imply that $\partial^L_i(T_p(Y))\in\mathcal{S}^+(r_p(\chi))'$. This completes the proof of the theorem. □

6.3. *Coxeter relations between Lusztig isomorphisms*

The aim of this subsection is to prove Theorem 6.19, that is, Lusztig isomorphisms satisfy Coxeter type relations. Note that a case by case proof as in [Lus93, Subsection 33.2] is not reasonable because of the presence of dozens of different examples of rank 2.

In the following claims we will use the following setting.

Setting 6.12. Let $\chi\in\mathcal{X}$. Assume that χ' is p-finite for all $p\in I$, $\chi'\in\mathcal{G}(\chi)$. Let $i,j\in I$ with $i\neq j$. Let $i_{2n+1}=i$ and $i_{2n}=j$ for all $n\in\mathbb{Z}$. Let $M=|R^\chi_+\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|$.

Lemma 6.13. *Assume Setting* 6.12. *Then*

$$\begin{aligned} M&=\min\bigl\{m\in\mathbb{N}_0\bigm|\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_j)\in-\mathbb{N}^I_0\bigr\}\\ &=1+\min\bigl\{m\in\mathbb{N}_0\bigm|\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_j)=\alpha_{i_{m+1}}\bigr\}.\end{aligned}$$

Proof. See [HY08, Lemma 6]. The right-hand side has to be interpreted as ∞ if the minimum is taken over the empty set. □

The main result in this subsection is based on the following lemma.

Lemma 6.14. *Assume Setting* 6.12. *Let $m,r\in\mathbb{N}_0$, and assume that*

$$\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_i+r\alpha_j)=\alpha_{i_{m+1}}. \tag{6.9}$$

Then there exists $t\in\mathbb{N}_0$ such that $\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_j)=\alpha_{i_m}+t\alpha_{i_{m+1}}$.

Proof. By the definition of $\sigma^{\chi'}_k$, where $k\in\{i,j\}$, $\chi'\in\mathcal{G}(\chi)$, one obtains that $\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_j)\in\mathbb{Z}\alpha_i+\mathbb{Z}\alpha_j$, that is

$$\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_j)=t_0\alpha_{i_m}+t\alpha_{i_{m+1}} \tag{6.10}$$

for some $t_0,t\in\mathbb{Z}$. One has to show that $t_0=1$ and $t\in\mathbb{N}_0$. Let $B_1=\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}|_{\mathbb{Z}\alpha_i+\mathbb{Z}\alpha_j}$ and let $B=B_1\circ B_2$, where $B_2\in\operatorname{End}(\mathbb{Z}\alpha_i+\mathbb{Z}\alpha_j)$ with $B_2(\alpha_i)=\alpha_i+r\alpha_j$, $B_2(\alpha_j)=\alpha_j$. Eqs. (6.9), (6.10) imply that B maps α_i to $\alpha_{i_{m+1}}$ and α_j to $t_0\alpha_{i_m}+t\alpha_{i_{m+1}}$. By Eq. (3.10), $\det\sigma^{\chi'}_k|_{\mathbb{Z}\alpha_i\oplus\mathbb{Z}\alpha_j}=-1$ for all $k\in\{i,j\}$, $\chi'\in\mathcal{G}(\chi)$, and hence $\det B=(-1)^m$. Hence $t_0=1$. Axioms (R1) and (R3) imply that $t\in\mathbb{N}_0$. □

Proposition 6.15. *Assume Setting* 6.12. *Let $m,r\in\mathbb{N}_0$. Assume that*

$$m<M,\quad \sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}(\alpha_i+r\alpha_j)\in\mathbb{N}^I_0. \tag{6.11}$$

Let $w=\sigma_{i_m}\cdots\sigma_{i_2}\sigma^\chi_{i_1}$. Then for $E^+_{i,r(j)},E^-_{i,r(j)}\in U(\chi)$

$$T_{i_m}\cdots T_{i_2}T_{i_1}\bigl(E^+_{i,r(j)}\bigr)\in U^+\bigl(w^*\chi\bigr)_{w(\alpha_i+r\alpha_j)}, \tag{6.12}$$

$$T^-_{i_m}\cdots T^-_{i_2}T^-_{i_1}\bigl(E^-_{i,r(j)}\bigr)\in U^+\bigl(w^*\chi\bigr)_{w(\alpha_i+r\alpha_j)}. \tag{6.13}$$

In particular, if $w(\alpha_i + r\alpha_j) = \alpha_{i_{m+1}}$*, then*

$$T_{i_m} \cdots T_{i_1}\big(\Bbbk E^+_{i,r(j)}\big) = \Bbbk E_{i_{m+1}}, \qquad T^-_{i_m} \cdots T^-_{i_1}\big(\Bbbk E^-_{i,r(j)}\big) = \Bbbk E_{i_{m+1}}. \tag{6.14}$$

Proof. The last statement of the proposition follows at once from the equation $U^+(w^*\chi)_{\alpha_i} = \Bbbk E_i$ and the fact that the maps T_p, where $p \in \{i, j\}$, are algebra isomorphisms.

The remaining assertions will be proven by induction on m. If $m = 0$, then the claim follows from the definition of $E^{\pm}_{i,r(j)}$. Assume now that $m > 0$ and that the lemma holds for all smaller values of m. First we prove by an indirect proof that $r > 0$. Assume that $r = 0$. Then

$$R^{w^*\chi}_+ \ni \sigma_{i_m} \cdots \sigma_{i_2}\sigma^{\chi}_{i_1}(\alpha_i) = \sigma_{i_m} \cdots \sigma_{i_3}\sigma^{r_1(\chi)}_{i_2}(-\alpha_i) \in -\mathbb{N}_0\alpha_i - \mathbb{N}_0\alpha_j,$$

where the last relation follows from Lemma 6.13 and the first formula of assumption (6.11). The obtained relation

$$R^{w^*\chi}_+ \cap -(\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) \neq \emptyset$$

is a contradiction to $R^{w^*\chi}_+ \subset \mathbb{N}^I_0$, and hence $r > 0$.

Now we perform the induction step by induction on r. One gets

$$T_{i_m} \cdots T_{i_2}T_{i_1}\big(E^+_{i,r(j)}\big) = T_{i_m} \cdots T_{i_2}T_{i_1}\big(E_jE^+_{i,r-1(j)} - \big(K_j \cdot E^+_{i,r-1(j)}\big)E_j\big).$$

By Theorem 6.11 this is equal to

$$\begin{aligned} &T_{i_m} \cdots T_{i_2}T_{i_1}(E_j)T_{i_m} \cdots T_{i_2}T_{i_1}\big(E^+_{i,r-1(j)}\big) \\ &\quad - \big(T_{i_m} \cdots T_{i_2}T_{i_1}(K_j) \cdot T_{i_m} \cdots T_{i_2}T_{i_1}\big(E^+_{i,r-1(j)}\big)\big)T_{i_m} \cdots T_{i_2}T_{i_1}(E_j). \end{aligned}$$

If $w(\alpha_i + (r-1)\alpha_j) \in R^{w^*\chi}_+$, then after replacing $T_{i_1}(E_j) = T_i(E_j)$ by $E^+_{j,-c_{ij}(i)}$ in the above formula one can apply the induction hypotheses for $m-1$ respectively $r-1$. Thus in this case relation (6.12) holds.

Assume now that $w(\alpha_i + (r-1)\alpha_j) \notin R^{w^*\chi}_+$. In this case, which covers the case $r = 1$, we will not use induction hypothesis on r. This way we ensure that the basis of the induction will be proven.

By [HY08, Lemma 1] there exists $n \in \mathbb{N}_0$ with $n < m$ such that $\sigma_{i_n} \cdots \sigma_{i_2}\sigma^{\chi}_{i_1}(\alpha_i + (r-1)\alpha_j) = \alpha_{i_{n+1}}$. Therefore induction hypothesis (on m) gives that

$$\begin{aligned} T_{i_m} \cdots T_{i_2}T_{i_1}\big(\Bbbk E^+_{i,r(j)}\big) = \Bbbk\big(&T_{i_m} \cdots T_{i_2}T_{i_1}(E_j)T_{i_m} \cdots T_{i_{n+2}}T_{i_{n+1}}(E_{i_{n+1}}) \\ &- \big(T_{i_m} \cdots T_{i_2}T_{i_1}(K_j) \cdot T_{i_m} \cdots T_{i_{n+2}}T_{i_{n+1}}(E_{i_{n+1}})\big)T_{i_m} \cdots T_{i_2}T_{i_1}(E_j)\big). \end{aligned}$$

By Lemma 6.14, $\sigma_{i_n} \cdots \sigma_{i_2}\sigma^{\chi}_{i_1}(\alpha_j) = \alpha_{i_n} + t\alpha_{i_{n+1}}$ for some $t \in \mathbb{N}_0$. Since $n < m$, the second formula in Eq. (6.14) together with relations $T_pT^-_p = \mathrm{id}$ for all $p \in I$ imply that

$$\begin{aligned} T_{i_m} \cdots T_{i_2}T_{i_1}\big(\Bbbk E^+_{i,r(j)}\big) = \Bbbk T_{i_m} \cdots T_{i_{n+1}}\big(&E^-_{i_n,t(i_{n+1})}E_{i_{n+1}} \\ &- \big(K_{i_n}K^t_{i_{n+1}} \cdot E_{i_{n+1}}\big)E^-_{i_n,t(i_{n+1})}\big). \end{aligned}$$

Using Lemma 4.22 and Lemma 6.7 one gets

$$\begin{aligned} T_{i_m} \cdots T_{i_2} T_{i_1}\big(\Bbbk E^+_{i,r(j)}\big) &= \Bbbk T_{i_m} \cdots T_{i_{n+1}}\big(E^-_{i_n,t+1(i_{n+1})}\big) \\ &= \Bbbk T_{i_m} \cdots T_{i_{n+2}}\big(E^+_{i_n,t'(i_{n+1})}\big) \end{aligned}$$

for some $t' \in \mathbb{N}_0$. Now one has $m-n-1<m$, and hence induction hypothesis can be applied to the last formula to obtain the statement of the lemma for $E^+_{i,r(j)}$.

The proof of the induction step for $E^-_{i,r(j)}$ goes analogously. $\square$

Corollary 6.16. *Assume Setting* 6.12. *Let $m \in \mathbb{N}_0$. Let $w = \sigma_{i_m} \cdots \sigma_{i_2} \sigma^\chi_{i_1}$.*

(i) *If $m < M$, then for $E_j \in U(\chi)$*

$$T_{i_m} \cdots T_{i_2} T_{i_1}(E_j) \in U^+\big(w^*\chi\big)_{w(\alpha_j)}, \tag{6.15}$$

$$T^-_{i_m} \cdots T^-_{i_2} T^-_{i_1}(E_j) \in U^+\big(w^*\chi\big)_{w(\alpha_j)}. \tag{6.16}$$

(ii) *If $m = M$ then*

$$\Bbbk T_{i_m} \cdots T_{i_2} T_{i_1}(E_j) = \Bbbk F_{i_m} L^{-1}_{i_m} = \Bbbk T_{i_{m-1}} \cdots T_{i_1} T_{i_0}(E_j). \tag{6.17}$$

Proof. For (i) use that

$$T_{i_1}(E_j) = E^+_{j,-c_{ij}(i)}, \qquad T^-_{i_1}(\Bbbk E_j) = \Bbbk E^-_{j,-c_{ij}(i)}$$

and apply Proposition 6.15. The first equation of (ii) follows from Eq. (6.15) with $m = M-1$ and from equation $\sigma_{i_{m-1}} \cdots \sigma_{i_2} \sigma^\chi_{i_1}(\alpha_j) = \alpha_{i_m}$.

Recall that $\phi_4^2 = \mathrm{id}$ by Proposition 4.12(iv). Thus, by Proposition 6.8(ii), for all $p \in I$ there exists $\underline{a} \in (\Bbbk^\times)^I$ such that $\phi_4 T_p = T^-_p \phi_4 \varphi_{\underline{a}}$. Therefore, via multiplication from the left with ϕ_4, the second equation in (ii) is equivalent to

$$\Bbbk L^{-1}_{i_m} E_{i_m} = \Bbbk T^-_{i_{m-1}} \cdots T^-_{i_1} T^-_{i_0}(F_j).$$

The right-hand side of this equation is equal to $\Bbbk T^-_{i_{m-1}} \cdots T^-_{i_1}(L^{-1}_j E_j)$, and hence the claim follows from (i) with $m = M-1$. $\square$

Lemma 6.17. *Assume Setting* 6.12. *Let $k \in I \setminus \{i, j\}$ and $m \in \mathbb{N}_0$. Let $w_m = \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma^\chi_{i_m}$. If $m \leqslant M$, then*

$$T_{i_1} \cdots T_{i_m}(E_k) \in U^+_{+i}\big(w^*_m\chi\big) \cap U^+_{+j}\big(w^*_m\chi\big),$$

where $E_k \in U(\chi)$. If $m < M$, then

$$T_{i_1} \cdots T_{i_m}(E_k) \in U^+_{-j}\big(w^*_m\chi\big). \tag{6.18}$$

Proof. We proceed by induction on m. For $m = 0$ the lemma clearly holds. Let now $m > 0$. Then the relation

$$T_{i_1} \cdots T_{i_m}(E_k) = T_{i_1}\big(T_{i_2} \cdots T_{i_m}(E_k)\big) \in U^+_{+i}\big(w^*_m\chi\big)$$

follows immediately from Eq. (6.18) with i and j interchanged and from Lemma 6.7(d). According to Lemma 4.15 and Proposition 5.10 it remains to show that

$$\big[F_j, T_{i_1}\cdots T_{i_m}(E_k)\big]=0 \qquad \text{if } m<M, \tag{6.19}$$

$$\big[F_j, T_{i_1}\cdots T_{i_m}(E_k)\big]\in L_jU^+\big(w_m^*\chi\big) \quad \text{if } m=M<\infty. \tag{6.20}$$

By the first equation in Proposition 6.8(ii),

$$\big[F_j, T_{i_1}\cdots T_{i_m}(E_k)\big]=T_{i_1}\cdots T_{i_m}\big[T^-_{i_m}\cdots T^-_{i_1}(F_j), E_k\big].$$

If $m<M$, then Corollary 6.16 and Proposition 6.8 imply that the expression $T^-_{i_m}\cdots T^-_{i_1}(F_j)$ lies in the subalgebra of $U^-(\chi)$ generated by F_i and F_j. Thus the above commutator is zero and hence Eq. (6.19) holds. On the other hand, if $m=M$, then

$$T^-_{i_m}\cdots T^-_{i_1}(\Bbbk F_j)=T^-_{i_m}(\Bbbk F_{i_m})=\Bbbk E_{i_m}L^{-1}_{i_m}$$

and hence

$$\begin{aligned}\Bbbk\big[F_j, T_{i_1}\cdots T_{i_m}(E_k)\big]&=\Bbbk T_{i_1}\cdots T_{i_m}\big[E_{i_m}L^{-1}_{i_m}, E_k\big]\\ &=\Bbbk T_{i_1}\cdots T_{i_m}\big(L^{-1}_{i_m}\big)T_{i_1}\cdots T_{i_m}\big(E^-_{k,1(i_m)}\big)\\ &=\Bbbk T_{i_1}\cdots T_{i_{m-1}}(L_{i_m})T_{i_1}\cdots T_{i_{m-1}}\big(E^+_{k,t(i_m)}\big)\end{aligned}$$

for some $t\in\mathbb{N}_0$. Since $\sigma_{i_1}\cdots\sigma_{i_{m-2}}\sigma_{i_{m-1}}^{r_{i_m}(\chi)}(\alpha_{i_m})=\alpha_j$, induction hypothesis and Corollary 6.16 imply that Eq. (6.20) holds for $m=M$. $\square$

Lemma 6.18. *Assume Setting* 6.12. *Let $k\in I\setminus\{i,j\}$ and $m\in\mathbb{N}_0$. Assume that $m<M<\infty$. Then*

$$\begin{aligned}T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)&\in U^+_{+i}\big(r_{i_1}\cdots r_{i_{m+M-1}}r_{i_{m+M}}(\chi)\big)\\ &\quad\cap U^+_{+j}\big(r_{i_1}\cdots r_{i_{m+M-1}}r_{i_{m+M}}(\chi)\big),\end{aligned}$$

where $E_k\in U(\chi)$. Further, if $m>0$, then

$$T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\in U^+_{-i}\big(r_{i_1}\cdots r_{i_{m+M-1}}r_{i_{m+M}}(\chi)\big).$$

Proof. If $m=0$, then the lemma holds by Lemma 6.17. Suppose now that $0<m<M$. Then by Proposition 5.10 and Lemma 4.15 it suffices to show that the following relations hold.

$$T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\in U^+\big(r_{i_1}\cdots r_{i_{m+M-1}}r_{i_{m+M}}(\chi)\big), \tag{6.21}$$

$$\big[F_i, T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big]=0, \tag{6.22}$$

$$\partial^K_j\big(T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big)=0. \tag{6.23}$$

We proceed by induction on m. Induction hypothesis gives that

$$T^-_{i_2}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\in U^+_{+i}\big(r_{i_2}\cdots r_{i_{m+M-1}}r_{i_{m+M}}(\chi)\big).$$

Thus Lemma 6.7(d) implies Eq. (6.21). For Eq. (6.22) one calculates

$$\begin{aligned}
&\Bbbk\big[F_i, T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big]\\
&\quad=\Bbbk T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}\big(\big[T^-_{i_{m+M}}\cdots T^-_{i_{m+1}}T_{i_m}\cdots T_{i_1}(F_i), E_k\big]\big)\\
&\quad=\Bbbk T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}\big(\big[T^-_{i_{m+M}}\cdots T^-_{i_{m+1}}T_{i_m}\cdots T_{i_2}\big(K_i^{-1}E_i\big), E_k\big]\big).
\end{aligned}$$

Corollary 6.16(i) with i and j interchanged implies that $x_m := T_{i_m}\cdots T_{i_2}(E_i)$ is a $\mathbb{Z}^I$-homogeneous element of $U^+(r_{i_{m+1}}\cdots r_{i_{m+M}}(\chi))$, and its degree is an element of $\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j$. Since $T^-_{i_{m+M}}\cdots T^-_{i_{m+1}}$ is an algebra map, the expressions $T^-_{i_{m+M}}\cdots T^-_{i_{m+1}}(x_m)$ and $T^-_{i_{m+M-1}}\cdots T^-_{i_m}(x_m)$ coincide up to a non-zero constant factor by Corollary 6.16(ii). (More precisely, one has to use a version of Corollary 6.16 with T_i^- and T_j^-, but this follows from Eq. (6.17) using manipulations with ϕ_4 as in the proof of Corollary 6.16(i).) One obtains the equations

$$\begin{aligned}
&\Bbbk\big[F_i, T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big]\\
&\quad=\Bbbk T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}\big(\big[T^-_{i_{m+M-1}}\cdots T^-_{i_m}T_{i_m}\cdots T_{i_2}\big(K_i^{-1}E_i\big), E_k\big]\big)\\
&\quad=\Bbbk T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}\big(\big[T^-_{i_{m+M-1}}\cdots T^-_{i_m}T_{i_m}\cdots T_{i_1}(F_i), E_k\big]\big)\\
&\quad=\Bbbk T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}\big(\big[T^-_{i_{M-m-1}}\cdots T^-_{i_0}(F_i), E_k\big]\big).
\end{aligned}$$

Since $m\geqslant 1$, one gets $M-m<M$. Thus Corollary 6.16 and Proposition 6.8 give that $T^-_{i_{M-m-1}}\cdots T^-_{i_0}(F_i)$ is in the subalgebra of $U(r_{i_{M-m-1}}\cdots r_{i_1}r_{i_0}(\chi))$ generated by F_i and F_j. Therefore the above commutator is zero and Eq. (6.22) is proven.

Eq. (6.23) can be obtained from Proposition 6.10(ii) as follows.

$$\begin{aligned}
&\Bbbk\partial^K_j\big(T^-_{i_1}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big)\\
&\quad=\Bbbk T^-_i\big(\big(\partial^K_i\big)^t\partial^K_j\big(T^-_{i_2}\cdots T^-_{i_m}T_{i_{m+1}}\cdots T_{i_{m+M}}(E_k)\big)\big)=0,
\end{aligned}$$

where $t\in\mathbb{N}_0$ is an appropriate integer and the last equation follows from Proposition 5.10 and the induction hypothesis. □

Theorem 6.19. *Let $\chi\in\mathcal{X}$. Assume that χ' is p-finite for all $p\in I$, $\chi'\in\mathcal{G}(\chi)$. Let $i,j\in I$ with $i\neq j$. Let $i_{2n+1}=i$ and $i_{2n}=j$ for all $n\in\mathbb{Z}$. Assume that $M=|R^\chi_+\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|<\infty$. Then there exists $\underline{a}\in(\Bbbk^\times)^I$ such that*

$$T_{i_M}\cdots T_{i_2}T_{i_1}=T_{i_{M-1}}\cdots T_{i_1}T_{i_0}\varphi_{\underline{a}}\tag{6.24}$$

as algebra isomorphisms $U(\chi)\to U(r_{i_M}\cdots r_{i_2}r_{i_1}(\chi))$.

Proof. By Proposition 6.8 the statement of the theorem is equivalent to

$$T_{i_M}\cdots T_{i_2}T_{i_1}(\Bbbk E_k)=T_{i_{M-1}}\cdots T_{i_1}T_{i_0}(\Bbbk E_k)\quad\text{for all } k\in I.\tag{6.25}$$

By Corollary 6.16 the above equation is fulfilled for $k\in\{i,j\}$. Suppose now that $k\notin\{i,j\}$. Then Lemma 6.18 (for $m=M-1$ and i,j interchanged) and Lemma 6.7(d) imply that

$$T^-_{i_1}\cdots T^-_{i_M}T_{i_{M+1}}\cdots T_{i_{2M}}(E_k)\in U^+\big(r_{i_1}\cdots r_{i_{2M-1}}r_{i_{2M}}(\chi)\big).$$

By Theorem 3.6, $\sigma_{i_1}\cdots\sigma_{i_{2M-1}}\sigma^{\chi}_{i_{2M}}=\mathrm{id}$. Hence

$$T^-_{i_1}\cdots T^-_{i_M}T_{i_{M+1}}\cdots T_{i_{2M}}(E_k)\in U^+(\chi)_{\alpha_k},$$

and therefore $T^-_{i_1}\cdots T^-_{i_M}T_{i_{M+1}}\cdots T_{i_{2M}}(E_k)\in \Bbbk^\times E_k$. Using equations $T_pT^-_p=\mathrm{id}$ from Proposition 6.8(ii), where $p\in\{i,j\}$, one gets Eq. (6.25) for $k\notin\{i,j\}$. $\square$

Theorem 6.20. *Let $\chi\in\mathcal{X}$. Assume that χ' is i-finite for all $i\in I$, $\chi'\in\mathcal{G}(\chi)$. Let $m\in\mathbb{N}_0$ and $i_1,\dots,i_m\in I$ such that $w=\sigma_{i_m}\cdots\sigma_{i_2}\sigma^{\chi}_{i_1}\in\mathrm{Hom}(\mathcal{W}(\chi))$ is a reduced expression. Let $p\in I$. Assume that $w(\alpha_p)\in R^{w^*\chi}_+$. Then*

$$T_{i_m}\cdots T_{i_1}(E_p)\in U^+\big(w^*\chi\big), \tag{6.26}$$

where $E_p\in U(\chi)$.

Proof. We proceed by induction on m. If $m=0$, then there is nothing to prove. If $m=1$, then $i_1\neq p$ by assumption and hence the theorem holds by definition of T_{i_1}.

Assume now that $m\geqslant 2$ and that the theorem is true for all smaller values of m. Then again $p\neq i_1$ by [HY08, Corollary 3]. Let $j_{2n}=p$ and $j_{2n+1}=i_1$ for all $n\in\mathbb{N}_0$. Let $r\in\mathbb{N}_0$ be maximal with respect to the property that $\ell(w_r)=m-r$, where $w_r=w\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_r}$. Then $1\leqslant r\leqslant m$, since $w\sigma_{j_1}=\sigma_{i_m}\cdots\sigma_{i_3}\sigma^{r_{i_1}(\chi)}_{i_2}$. Further, $\ell(w_r\sigma_{j_{r+1}})=m-r+1$ by the maximality of r and $\ell(w_r\sigma_{j_r})=m-r+1$ since $w_r\sigma_{j_r}=w_{r-1}$. Let $k_1,\dots,k_{m-r}\in I$ such that

$$w_r=\sigma_{k_{m-r}}\cdots\sigma_{k_2}\sigma^{(w'_r)^*\chi}_{k_1},\quad \text{where } w'_r=\sigma_{j_r}\cdots\sigma_{j_2}\sigma^{\chi}_{j_1}.$$

Then $w=w_rw'_r$, and hence Theorem 6.19 and Matsumoto's theorem, see [HY08, Theorem 5], imply that

$$T_{k_{m-r}}\cdots T_{k_1}T_{j_r}\cdots T_{j_1}=T_{i_m}\cdots T_{i_2}T_{i_1}\varphi_{\underline{a}}$$

for some $\underline{a}\in(\Bbbk^\times)^I$. Further, the assumption $w(\alpha_p)\in R^{w^*\chi}_+$ implies that $\sigma_{j_r}\cdots\sigma_{j_2}\sigma^{\chi}_{j_1}(\alpha_p)\in R^{(w'_r)^*\chi}_+$, and hence $T_{j_r}\cdots T_{j_1}(E_p)$ lies in the subalgebra of $U^+((w'_r)^*\chi)$ generated by E_p and E_{i_1}. Since $m-r<m$, induction hypothesis implies that

$$T_{k_{m-r}}\cdots T_{k_1}(E_{k_0})\in U^+\big(w^*\chi\big)\quad \text{for } k_0\in\{p,i_1\}.$$

Since $T_{k_{m-r}}\cdots T_{k_1}$ is an algebra map, Eq. (6.26) holds for m. $\square$

Recall the algebra map $\varphi_\tau: U(\chi)\to U(\tau^*\chi)$ defined in Proposition 4.9, where τ is a permutation of I. Let $\hat\tau$ be the automorphism of $\mathbb{Z}^I$ given by $\hat\tau(\alpha_p)=\alpha_{\tau(p)}$ for all $p\in I$. Note that for any $\chi\in\mathcal{X}$ and all $i,j\in I$ one has $\chi(-\alpha_i,-\alpha_j)=\chi(\alpha_i,\alpha_j)$, and hence $(-\mathrm{id})^*\chi=\chi$. Further, if χ' is p-finite for all $\chi'\in\mathcal{G}(\chi)$ and $p\in I$, then for each $\chi'\in\mathcal{G}(\chi)$ there is a unique longest element $w_0\in\mathrm{Hom}(\chi',\_)\subset\mathrm{Hom}(\mathcal{W}(\chi))$, see [HY08, Corollary 5].

Corollary 6.21. *Let $\chi\in\mathcal{X}$. Assume that $M=|R^{\chi}_+|$ is finite. Let $i_1,\dots,i_M\in I$ such that $w_0=\sigma_{i_M}\cdots\sigma_{i_2}\sigma^{\chi}_{i_1}$ is a longest element of* $\mathrm{Hom}(\mathcal{W}(\chi))$. *Then there exists $\underline{\lambda}\in(\Bbbk^\times)^I$ and a permutation τ of I such that $w_0=-\hat\tau$ and*

$$T_{i_M}\cdots T_{i_2}T_{i_1}=\phi_1\circ\varphi_\tau\circ\varphi_{\underline{\lambda}}$$

as algebra maps $U(\chi)\to U(w_0^*\chi)$.

Proof. Since $w_0(R^\chi_+) = -R^{w_0^*\chi}_+$, there exists a unique permutation τ of I such that $w_0(\alpha_i) = -\alpha_{\tau(i)}$ for all $i \in I$.

Let $p \in I$. Since w_0 has maximal length, $\ell(\sigma_p w_0) = M - 1$. Let $j_1, \dots, j_{M-1} \in I$ such that $\sigma_p \sigma_{j_{M-1}} \cdots \sigma_{j_2} \sigma^\chi_{j_1} = w_0$. By Theorem 6.19

$$T_{w_0} := T_{i_M} \cdots T_{i_1} = T_p T_{j_{M-1}} \cdots T_{j_1} \varphi_{\underline{a}}$$

for some $\underline{a} \in (\Bbbk^\times)^I$. Further, Theorem 6.20 and relation $\sigma_p w_0(\alpha_{\tau^{-1}(p)}) = \alpha_p$ imply that there exists $\lambda_{\tau^{-1}(p)} \in \Bbbk^\times$ such that

$$T_{j_{M-1}} \cdots T_{j_1} \varphi_{\underline{a}}(E_{\tau^{-1}(p)}) = \lambda_{\tau^{-1}(p)} E_p.$$

Thus

$$\begin{aligned} T_{w_0}(E_{\tau^{-1}(p)}) &= T_p T_{j_{M-1}} \cdots T_{j_1} \varphi_{\underline{a}}(E_{\tau^{-1}(p)}) \\ &= \lambda_{\tau^{-1}(p)} T_p(E_p) = \lambda_{\tau^{-1}(p)} F_p L_p^{-1}. \end{aligned}$$

Put $\underline{\lambda} = (\lambda_i)_{i \in I}$. Similarly to the above arguments one can show that for each $i \in I$ there exists $\mu_i \in \Bbbk^\times$ such that

$$T_{w_0}(K_i) = \phi_1\big(\varphi_\tau\big(\varphi_{\underline{\lambda}}(K_i)\big)\big), \qquad T_{w_0}(L_i) = \phi_1\big(\varphi_\tau\big(\varphi_{\underline{\lambda}}(L_i)\big)\big),$$
$$T_{w_0}(F_i) = \mu_i \phi_1\big(\varphi_\tau\big(\varphi_{\underline{\lambda}}(F_i)\big)\big).$$

Since T_{w_0} is an algebra map, one obtains that $\mu_i = 1$ for all $i \in I$. This proves the corollary. $\square$

7. A characterization of Nichols algebras of diagonal type

The following theorem, which is an application of the Lusztig isomorphisms constructed in the previous section, gives a characterization of Nichols algebras of diagonal type with finite root system.

Theorem 7.1. *Let $\chi \in \mathcal{X}$. Assume that R^χ_+ is finite. For each $\chi' \in \mathcal{G}(\chi)$ let $\mathcal{J}^+(\chi')$ be an ideal of $\mathcal{U}^+(\chi')$ such that*

$$\varepsilon\big(\mathcal{J}^+(\chi')\big) = \{0\}, \qquad X \cdot \mathcal{J}^+(\chi') \subset \mathcal{J}^+(\chi'), \qquad \mathcal{I}^+_p(\chi') \subset \mathcal{J}^+(\chi'),$$
$$\partial^K_p\big(\mathcal{J}^+(\chi')\big) \subset \mathcal{J}^+(\chi'), \qquad \partial^L_p\big(\mathcal{J}^+(\chi')\big) \subset \mathcal{J}^+(\chi')$$

for all $X \in \mathcal{U}^0(\chi')$, $p \in I$. For all $\chi' \in \mathcal{G}(\chi)$ let $\mathcal{J}(\chi')$ and $\tilde{\mathcal{J}}(\chi')$ be the ideals of $\mathcal{U}(\chi')$ generated by $\mathcal{J}^+(\chi') + \phi_4(\mathcal{J}^+(\chi'))$ and $\mathcal{J}^+(\chi') + \phi_4(\mathcal{S}^+(\chi'))$, respectively. The following are equivalent.

(1) *$\mathcal{U}^+(\chi')/\mathcal{J}^+(\chi') = U^+(\chi')$ for all $\chi' \in \mathcal{G}(\chi)$.*
(2) *$\mathcal{J}^+(\chi') = \mathcal{S}^+(\chi')$ for all $\chi' \in \mathcal{G}(\chi)$.*
(3) *The algebra maps $T_p : \mathcal{U}(\chi') \to \mathcal{U}(r_p(\chi'))/\mathcal{J}(r_p(\chi'))$ satisfy*

$$T_p\big(\mathcal{J}^+(\chi')\big) = \{0\} \quad \text{for all } \chi' \in \mathcal{G}(\chi),\ p \in I. \tag{7.1}$$

(4) *The algebra maps $T_p : \mathcal{U}(\chi') \to \mathcal{U}(r_p(\chi'))/\tilde{\mathcal{J}}(r_p(\chi'))$ satisfy*

$$T_p\big(\tilde{\mathcal{J}}(\chi')\big) = \{0\} \quad \text{for all } \chi' \in \mathcal{G}(\chi),\ p \in I. \tag{7.2}$$

(5) *The algebra maps* $T_p^- : \mathcal{U}(\chi') \to \mathcal{U}(r_p(\chi'))/\mathcal{J}(r_p(\chi'))$ *satisfy*

$$T_p^-\big(\mathcal{J}^+\big(\chi'\big)\big) = \{0\} \quad \textit{for all } \chi' \in \mathcal{G}(\chi),\ p \in I. \tag{7.3}$$

(6) *The algebra maps* $T_p^- : \mathcal{U}(\chi') \to \mathcal{U}(r_p(\chi'))/\tilde{\mathcal{J}}(r_p(\chi'))$ *satisfy*

$$T_p^-\big(\tilde{\mathcal{J}}\big(\chi'\big)\big) = \{0\} \quad \textit{for all } \chi' \in \mathcal{G}(\chi),\ p \in I. \tag{7.4}$$

If the statements in Theorem 7.1 are fulfilled, then because of Theorem 6.11 the algebra maps T_p, T_p^- in statements (3) and (5) induce isomorphisms $\mathcal{U}(\chi')/\mathcal{J}(\chi') \to \mathcal{U}(r_p(\chi'))/\mathcal{J}(r_p(\chi'))$ for all $\chi' \in \mathcal{G}(\chi)$, $p \in I$.

Proof of Theorem 7.1. The equivalence of claims (1) and (2) is the definition of $U^+(\chi')$. The implications (2) ⇒ (3) and (2)⇒(5) have been proven in Theorem 6.11.

Next we prove the implication (3) ⇒ (4). Let $\chi' \in \mathcal{G}(\chi)$ and $p \in I$. Then $\mathcal{J}^+(\chi') \subset \mathcal{S}^+(\chi')$ by Proposition 5.4. Thus one has to show that the maps $T_p : \mathcal{U}(\chi') \to \mathcal{U}(r_p(\chi'))/\mathcal{J}(r_p(\chi'))$ satisfy

$$T_p\big(\phi_4\big(\mathcal{S}^+\big(\chi'\big)\big)\big) \subset \big(\phi_4\big(\mathcal{S}^+\big(r_p\big(\chi'\big)\big)\big) + \mathcal{J}\big(r_p\big(\chi'\big)\big)\big)/\mathcal{J}\big(r_p\big(\chi'\big)\big). \tag{7.5}$$

Equivalently, since $\phi_4(\mathcal{J}(r_p(\chi'))) = \mathcal{J}(r_p(\chi'))$, the last equation in Proposition 6.8(ii) and Lemma 5.2 imply that relation (7.5) is equivalent to

$$T_p^-\big(\mathcal{S}^+\big(\chi'\big)\big) \subset \big(\mathcal{S}^+\big(r_p\big(\chi'\big)\big) + \mathcal{J}\big(r_p\big(\chi'\big)\big)\big)/\mathcal{J}\big(r_p\big(\chi'\big)\big).$$

Further, by Lemma 6.9(ii) it suffices to check the following inclusions.

$$T_p^-\big(\mathcal{S}^+\big(\chi'\big) \cap \mathcal{U}^+_{+p}\big(\chi'\big)\big) \subset \big(\mathcal{S}^+\big(r_p\big(\chi'\big)\big) + \mathcal{J}\big(r_p\big(\chi'\big)\big)\big)/\mathcal{J}\big(r_p\big(\chi'\big)\big), \tag{7.6}$$

$$T_p^-\big(\mathcal{S}^+\big(\chi'\big) \cap \Bbbk[E_p]\big) = \{0\}. \tag{7.7}$$

Now relation (7.6) follows from Lemma 6.7(d) and Theorem 6.11. Finally, Eq. (7.7) is a consequence of Eq. (6.8) and the assumption $\mathcal{J}_p^+(\chi') \subset \mathcal{J}^+(\chi')$. Thus the implication (3) ⇒ (4) is proven.

We finish the proof of the theorem with showing the implication (4) ⇒ (2). The remaining open implication (6) ⇒ (2) can be proven in a similar way.

Let $\chi' \in \mathcal{G}(\chi)$. Since $R_+^{\chi'}$ is finite, there exists a longest element $w_0 \in \mathrm{Hom}(\chi', \_) \subset \mathrm{Hom}(\mathcal{W}(\chi))$. Let $M = |R_+^{\chi'}|$ and $i_1, \dots, i_M \in I$ such that $\sigma_{i_M} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi'}$ is a reduced expression of w_0. By the assumption of statement (4) the map $T_{w_0} := T_{i_M} \cdots T_{i_1} : \mathcal{U}(\chi') \to \mathcal{U}(w_0^* \chi')/\tilde{\mathcal{J}}(w_0^* \chi')$ is well defined and satisfies

$$T_{w_0}\big(\tilde{\mathcal{J}}\big(\chi'\big)\big) = \{0\}.$$

In particular, $w_0(R_+^{\chi'}) = -R_+^{w_0^* \chi'}$ implies that

$$T_{w_0}\big(\phi_4\big(\mathcal{S}^+\big(\chi'\big)\big)\big) = \{0\}. \tag{7.8}$$

Because of the relations $\mathcal{I}_p^+(\chi') + \phi_4(\mathcal{I}_p^+(\chi')) \subset \mathcal{J}(\chi')$ the result of Corollary 6.21 holds also for T_{w_0}, namely

$$T_{w_0} = \phi_1 \circ \varphi_\tau \circ \varphi_{\underline{\lambda}}$$

for some $\underline{\lambda} \in (\mathbb{k}^\times)^I$ and a permutation τ of I. Thus Eq. (7.8) gives that

$$\phi_1\big(\varphi_\tau\big(\varphi_{\underline{\lambda}}\big(\phi_4\big(\mathcal{S}^+(\chi')\big)\big)\big)\big) \subset \mathcal{J}^+\big(w_0^*\chi'\big)\mathcal{U}^0\big(w_0^*\chi'\big),$$

and hence $\mathcal{S}^+(w_0^*\chi') \subset \mathcal{J}^+(w_0^*\chi')$ by Proposition 4.12 and Lemma 5.5. This proves the implication $(4) \Rightarrow (2)$. $\square$

We are going to give an application of Theorem 7.1, see Example 7.4. Owing to the fact that the representation theory is not yet developed, for the proof a couple of technical formulas are used, which can be obtained by standard techniques.

Lemma 7.2. *Let $\chi \in \mathcal{X}$, $\mu \in \mathbb{Z}^I$, and $p \in I$. Then for all $m \in \mathbb{N}_0$ and all $X \in \mathcal{U}(\chi)_\mu$ and $Y \in \mathcal{U}(\chi)$ one has*

$$(\operatorname{ad} E_p)^m(XY) = \sum_{n=0}^{m} \chi(n\alpha_p, \mu)\binom{m}{n}_{q_{pp}} (\operatorname{ad} E_p)^{m-n}X \cdot (\operatorname{ad} E_p)^n Y.$$

Proof. The algebra $\mathcal{U}(\chi)$ is a module algebra with respect to the adjoint action ad of $\mathcal{U}(\chi)$, and hence

$$(\operatorname{ad} Z)(XY) = (\operatorname{ad} Z_{(1)})X \cdot (\operatorname{ad} Z_{(2)})Y \quad \text{for all } Z \in \mathcal{U}(\chi).$$

Then Remark 4.2, Lemma 4.23(i), and Eqs. (4.21) and (4.22) imply the claim. $\square$

Corollary 7.3. *Let $\chi \in \mathcal{X}$ and $p, i \in I$ such that $p \neq i$ and $q_{pp}^{-c_{pi}} q_{pi} q_{ip} = 1$. Then for any $\mathbb{Z}^I$-homogeneous element $Y \in (\mathcal{U}^+_{+p}(\chi) + \mathcal{I}^+_p(\chi))/\mathcal{I}^+_p(\chi)$ with $(\operatorname{ad} E_p)^{r+1}Y = 0$ for some $r \in \mathbb{N}_0$ one has*

$$\begin{aligned}
&(\operatorname{ad} E_p)^{-c_{pi}+r}\big(E_i Y - (K_i \cdot Y)E_i\big) \\
&\quad = \binom{-c_{pi}+r}{r}_{q_{pp}} q_{pi}^r \big(E^+_{i,-c_{pi}} \cdot (\operatorname{ad} E_p)^r Y \\
&\qquad - \big(K_i K_p^{-c_{pi}} \cdot (\operatorname{ad} E_p)^r Y\big)E^+_{i,-c_{pi}}\big).
\end{aligned}$$

Proof. The left adjoint action of $\mathcal{U}(\chi)$ induces an action on the algebra $(\mathcal{U}^+_{+p}(\chi) + \mathcal{I}^+_p(\chi))/\mathcal{I}^+_p(\chi)$. Thus Lemma 7.2, Eq. (4.21), and relations $(\operatorname{ad} E_p)^{1-c_{pi}}E_i = (\operatorname{ad} E_p)^{r+1}Y = 0$ give that

$$\begin{aligned}
(\operatorname{ad} E_p)^{-c_{pi}+r}(E_i Y) &= \binom{-c_{pi}+r}{-c_{pi}}_{q_{pp}} q_{pi}^r E^+_{i,-c_{pi}} \cdot (\operatorname{ad} E_p)^r Y, \\
(\operatorname{ad} E_p)^{-c_{pi}+r}\big((K_i \cdot Y)E_i\big) &= \binom{-c_{pi}+r}{r}_{q_{pp}} \big((\operatorname{ad} E_p)^r\big(K_p^{-c_{pi}} K_i \cdot Y\big)\big)E^+_{i,-c_{pi}}.
\end{aligned}$$

The condition on χ in the corollary gives the equation $q_{pi}^r K_i K_p^{-c_{pi}} E_p^r = E_p^r K_i K_p^{-c_{pi}}$ which implies the claim. $\square$

Example 7.4. It was proven already by Lusztig [Lus93, Theorem 33.1.3] that for quantized symmetrizable Kac–Moody algebras $U_q(\mathfrak{g})$, defined over the field $\mathbb{Q}(q)$, Serre-relations (the generators of the ideals $\mathcal{I}^+_p(\chi)$) are sufficient to define the ideal $\mathcal{S}^+(\chi)$. A careful choice of related results on Kac–Moody algebras leads to the proof of this statement even if q is not a root of 1, see [HK07]. Using twisting of Nichols algebras, see [AS02b, Proposition 3.9, Remark 3.10] one can show that the analogous statement holds for multiparameter quantizations of Kac–Moody algebras over fields of characteristic zero. In this example an easy application of Theorem 7.1 is demonstrated on multiparameter quantizations of semisimple Lie algebras. As an improvement compared to [Lus93] it is allowed that $\Bbbk$ is an arbitrary field.

Let $\chi \in \mathcal{X}$. Assume that R^χ_+ is finite, and that $(m)_{q_{ii}} \neq 0$ for all $m \in \mathbb{N}$, $i \in I$. Thus χ is of (finite) Cartan type, that is, there is a symmetrizable Cartan matrix $C = (c_{ij})_{i,j\in I}$ of finite type such that

$$q_{ii}^{-c_{ij}} q_{ij} q_{ji} = 1 \tag{7.9}$$

for all $i, j \in I$. In this case $C^{\chi'} = C$ for all $\chi' \in \mathcal{G}(\chi)$.

Theorem 7.1 characterizes $U^+(\chi)$ which is the upper triangular part of the multiparameter version of a Drinfel'd–Jimbo algebra. In the present setting it can be easily proven that the ideal $\mathcal{S}^+(\chi)$ is generated by the Serre relations, that is

$$\mathcal{S}^+(\chi) = \sum_{p\in I} \mathcal{I}^+_p(\chi). \tag{7.10}$$

Indeed, by Definition 6.1 and Theorem 7.1(3) $\Rightarrow$ (2) one has to check that

$$T_p\big((\operatorname{ad} E_i)^{1-c_{ij}} E_j\big) = 0 \quad \text{for all } i, j, p \in I \text{ with } i \neq j. \tag{7.11}$$

If $p = i$, then Eq. (7.11) follows from Lemmata 4.32 and 6.7(c). If $p \neq i$ and $p \neq j$, then one gets

$$T_p\big((\operatorname{ad} E_i)^{1-c_{ij}} E_j\big) = \big(\widetilde{\operatorname{ad}} T_p(E_i)\big)^{1-c_{ij}} T_p(E_j) = \big(\widetilde{\operatorname{ad}} E^+_{i,-c_{pi}}\big)^{1-c_{ij}} E^+_{j,-c_{pj}},$$

where

$$\big(\widetilde{\operatorname{ad}} T_p(E_i)\big) X = T_p(E_i) X - \big(K_i K_p^{-c_{pi}} \cdot X\big) T_p(E_i).$$

Thus equations $E^+_{i,1-c_{pi}} = E^+_{j,1-c_{pj}} = 0$ and Corollary 7.3, which has to be applied $1 - c_{ij}$ times, imply that

$$T_p\big((\operatorname{ad} E_i)^{1-c_{ij}} E_j\big) \in \Bbbk^\times (\operatorname{ad} E_p)^{-c_{pi}(1-c_{ij})-c_{pj}} \big((\operatorname{ad} E_i)^{1-c_{ij}} E_j\big) = \{0\}.$$

It remains to consider the case $j = p \neq i$. If $c_{ij} = 0$, then in all algebras $\mathcal{U}(\chi')$ with $\chi' \in \mathcal{G}(\chi)$ we have

$$E_i E_p - (K_i \cdot E_p) E_i = E_i E_p - (L_i \cdot E_p) E_i \in \Bbbk^\times \big(E_p E_i - (K_p \cdot E_i) E_p\big).$$

This case was considered below Eq. (7.11). Thus, since R^χ_+ is finite, it remains to consider the case

$$\min\{c_{pi}, c_{ip}\} \in \{-1, -2, -3\}, \qquad \max\{c_{pi}, c_{ip}\} = -1.$$

We are going to show that

$$\Bbbk T_p\big((\operatorname{ad} E_i)^{1-c_{ip}} E_p\big) = \Bbbk (\operatorname{ad} E_p)^{-c_{pi}(1-c_{ip})-2} \big(\operatorname{ad}' E_i\big)^{1-c_{ip}} E_p, \tag{7.12}$$

where $(\mathrm{ad}' E_i)X = E_iX - (L_i \cdot X)E_i$. In fact, ad' can be considered as the left adjoint action of $\mathcal{U}(\chi)$ on itself via a second Hopf algebra structure of $\mathcal{U}(\chi)$, but we will not use this structure. Further, Lemma 4.32 gives that the above equality finishes the proof of Eq. (7.10).

WARNING!!! Since χ is not symmetric, the structure constants of χ and $r_p(\chi)$ do not coincide. Without loss of generality we may assume that both sides of Eq. (7.12) are in $(\mathcal{U}^+_{+p}(\chi) + \mathcal{I}^+_p(\chi))/\mathcal{I}^+_p(\chi)$, and hence in both expressions we may use the structure constants of χ.

On the one hand we have

$$\begin{aligned}\Bbbk T_p\big((\mathrm{ad}\, E_i)^{1-c_{ip}} E_p\big) &= \Bbbk T_p\big((\mathrm{ad}\, E_i)^{-c_{ip}} E^-_{i,1}\big)\\ &= \Bbbk\big(\widetilde{\mathrm{ad}} E^+_{i,-c_{pi}}\big)^{-c_{ip}} E^+_{i,-c_{pi}-1}.\end{aligned}$$

For this we can give an explicit formula by performing in Eq. (4.45) the following replacements:

$$\begin{gathered}E_p \mapsto E^+_{i,-c_{pi}}, \qquad E_i \mapsto E^+_{i,-c_{pi}-1}, \qquad K_p \mapsto K_i K_p^{-c_{pi}},\\ q_{pp} \mapsto q_{ii}, \qquad q_{pi} \mapsto q_{ii}q_{pi}, \qquad m \mapsto -c_{ip}.\end{gathered}$$

One obtains that

$$\begin{aligned}&\Bbbk T_p\big((\mathrm{ad}\, E_i)^{1-c_{ip}} E_p\big)\\ &\quad= \Bbbk \sum_{s=0}^{-c_{ip}} (-q_{pi})^s q_{ii}^{s(s+1)/2} \binom{-c_{ip}}{s}_{q_{ii}} \big(E^+_{i,-c_{pi}}\big)^{-c_{ip}-s} E^+_{i,-c_{pi}-1} \big(E^+_{i,-c_{pi}}\big)^s.\end{aligned} \tag{7.13}$$

Let first $c_{ip} = -1$. Then $q_{ii}q_{ip}q_{pi} = 1$, and hence Lemma 7.2 yields that

$$\begin{aligned}&\Bbbk(\mathrm{ad}\, E_p)^{-2c_{pi}-2}\big(\mathrm{ad}' E_i\big)^2 E_p\\ &\quad= \Bbbk(\mathrm{ad}\, E_p)^{-2c_{pi}-2}\big(\mathrm{ad}' E_i\big) E^+_{i,1}\\ &\quad= \Bbbk(\mathrm{ad}\, E_p)^{-2c_{pi}-2}\big(E_i E^+_{i,1} - q_{ip} E^+_{i,1} E_i\big)\\ &\quad= \Bbbk\Bigg(\binom{-2c_{pi}-2}{-c_{pi}-1}_{q_{pp}} \big(q_{pi}^{-c_{pi}-1} E^+_{i,-c_{pi}-1} E^+_{i,-c_{pi}}\\ &\qquad - q_{ip} q_{pi}^{-c_{pi}-1} q_{pp}^{-c_{pi}-1} E^+_{i,-c_{pi}} E^+_{i,-c_{pi}-1}\big)\\ &\qquad + \binom{-2c_{pi}-2}{-c_{pi}}_{q_{pp}} \big(q_{pi}^{-c_{pi}-2} E^+_{i,-c_{pi}} E^+_{i,-c_{pi}-1}\\ &\qquad - q_{ip} q_{pi}^{-c_{pi}} q_{pp}^{-c_{pi}} E^+_{i,-c_{pi}-1} E^+_{i,-c_{pi}}\big)\Bigg).\end{aligned}$$

Using Lemma 3.1 this gives

$$\begin{aligned}&= \Bbbk\binom{-2c_{pi}-2}{-c_{pi}-1}_{q_{pp}} \frac{1}{(-c_{pi})_{q_{pp}}} \big(q_{pi}^{-c_{pi}-1} (-c_{pi})_{q_{pp}} E^+_{i,-c_{pi}-1} E^+_{i,-c_{pi}}\\ &\qquad - q_{ip} q_{pi}^{-c_{pi}-1} q_{pp}^{-c_{pi}-1} (-c_{pi})_{q_{pp}} E^+_{i,-c_{pi}} E^+_{i,-c_{pi}-1}\end{aligned}$$

$$
\begin{aligned}
&+ q_{pi}^{-c_{pi}-2}(-c_{pi}-1)_{q_{pp}} E^+_{i,-c_{pi}} E^+_{i,-c_{pi}-1} \\
&- q_{ip} q_{pi}^{-c_{pi}} q_{pp}^{-c_{pi}} (-c_{pi}-1)_{q_{pp}} E^+_{i,-c_{pi}-1} E^+_{i,-c_{pi}}\big).
\end{aligned}
$$

By Eq. (7.9) one has $q_{pp}^{-c_{pi}} q_{ip} q_{pi} = 1$, and hence we conclude that

$$
= \Bbbk\big(-q_{pi}^{-c_{pi}-2} q_{pp}^{-1} E^+_{i,-c_{pi}} E^+_{i,-c_{pi}-1} + q_{pi}^{-c_{pi}-1} q_{pp}^{-c_{pi}-1} E^+_{i,-c_{pi}-1} E^+_{i,-c_{pi}}\big).
$$

The latter formula coincides with the one in Eq. (7.13) if $c_{ip} = -1$.

Let now $c_{pi} = -1$ and $c_{ip} = -2$. Then

$$
\begin{aligned}
&(\operatorname{ad} E_p)\big(\operatorname{ad}' E_i\big)^2 E^+_{i,1} \\
&\quad = (\operatorname{ad} E_p)\big(\operatorname{ad}' E_i\big)\big(E_i E^+_{i,1} - q_{ii}^{-1} q_{pi}^{-1} E^+_{i,1} E_i\big) \\
&\quad = (\operatorname{ad} E_p)\big(E_i^2 E^+_{i,1} - \big(q_{ii}^{-1} + q_{ii}^{-2}\big) q_{pi}^{-1} E_i E^+_{i,1} E_i + q_{ii}^{-3} q_{pi}^{-2} E^+_{i,1} E_i^2\big) \\
&\quad = E^+_{i,1} E_i E^+_{i,1} + q_{pi} E_i \big(E^+_{i,1}\big)^2 \\
&\qquad - \big(q_{ii}^{-1} + q_{ii}^{-2}\big) q_{pi}^{-1} \big(E^+_{i,1}\big)^2 E_i - \big(q_{ii}^{-1} + q_{ii}^{-2}\big) q_{pi} q_{pp} E_i \big(E^+_{i,1}\big)^2 \\
&\qquad + q_{ii}^{-3} q_{pi}^{-1} q_{pp} \big(E^+_{i,1}\big)^2 E_i + q_{ii}^{-3} q_{pp} E^+_{i,1} E_i E^+_{i,1} \\
&\quad = -q_{ii}^{-2} q_{pi}^{-1} \big(E^+_{i,1}\big)^2 E_i + \big(1 + q_{ii}^{-1}\big) E^+_{i,1} E_i E^+_{i,1} - q_{ii} q_{pi} E_i \big(E^+_{i,1}\big)^2.
\end{aligned}
$$

Similarly, if $c_{pi} = -1$ and $c_{ip} = -3$, then

$$
\begin{aligned}
&(\operatorname{ad} E_p)^2\big(\operatorname{ad}' E_i\big)^3 E^+_{i,1} \\
&\quad = (\operatorname{ad} E_p)^2\big(E_i^3 E^+_{i,1} - \big(q_{ii}^{-1} + q_{ii}^{-2} + q_{ii}^{-3}\big) q_{pi}^{-1} E_i^2 E^+_{i,1} E_i \\
&\qquad + \big(q_{ii}^{-3} + q_{ii}^{-4} + q_{ii}^{-5}\big) q_{pi}^{-2} E_i E^+_{i,1} E_i^2 - q_{ii}^{-6} q_{pi}^{-3} E^+_{i,1} E_i^3\big) \\
&\quad = (\operatorname{ad} E_p)\big(E^+_{i,1} E_i^2 E^+_{i,1} + q_{pi} E_i E^+_{i,1} E_i E^+_{i,1} + q_{pi}^2 E_i^2 \big(E^+_{i,1}\big)^2 \\
&\qquad - (3)_{q_{ii}} q_{ii}^{-3} q_{pi}^{-1} \big(E^+_{i,1} E_i E^+_{i,1} E_i + q_{pi} E_i \big(E^+_{i,1}\big)^2 E_i + q_{pi}^3 q_{pp} E_i^2 \big(E^+_{i,1}\big)^2\big) \\
&\qquad + (3)_{q_{ii}} q_{ii}^{-5} q_{pi}^{-2} \big(\big(E^+_{i,1}\big)^2 E_i^2 + q_{pi}^2 q_{pp} E_i \big(E^+_{i,1}\big)^2 E_i + q_{pi}^3 q_{pp} E_i E^+_{i,1} E_i E^+_{i,1}\big) \\
&\qquad - q_{ii}^{-6} q_{pi}^{-3} \big(q_{pi} q_{pp} \big(E^+_{i,1}\big)^2 E_i^2 + q_{pi}^2 q_{pp} E^+_{i,1} E_i E^+_{i,1} E_i + q_{pi}^3 q_{pp} E^+_{i,1} E_i^2 E^+_{i,1}\big)\big) \\
&\quad = (\operatorname{ad} E_p)\big((2)_{q_{ii}} q_{ii}^{-5} q_{pi}^{-2} \big(E^+_{i,1}\big)^2 E_i^2 - \big(1 + (3)_{q_{ii}}\big) q_{ii}^{-3} q_{pi}^{-1} E^+_{i,1} E_i E^+_{i,1} E_i \\
&\qquad + \big(1 - q_{ii}^{-3}\big) E^+_{i,1} E_i^2 E^+_{i,1} + \big(1 - q_{ii}^{-3}\big) E_i \big(E^+_{i,1}\big)^2 E_i \\
&\qquad + \big(1 + (3)_{q_{ii}} q_{ii}^{-2}\big) q_{pi} E_i E^+_{i,1} E_i E^+_{i,1} - (2)_{q_{ii}} q_{ii} q_{pi}^2 E_i^2 \big(E^+_{i,1}\big)^2\big) \\
&\quad = \big(q_{ii} + q_{ii}^2 - 2 - q_{ii} - q_{ii}^2 + 1 - q_{ii}^{-3}\big) \big(E^+_{i,1}\big)^3 E_i \\
&\qquad + q_{pi} \big(q_{ii} + q_{ii}^2 + q_{ii}^3 - 1 + 2 + q_{ii}^{-1} + q_{ii}^{-2}\big) \big(E^+_{i,1}\big)^2 E_i E^+_{i,1}
\end{aligned}
$$

$$
\begin{aligned}
&+q_{pi}^2\left(-2q_{ii}^3-q_{ii}^4-q_{ii}^5+q_{ii}^3-1-q_{ii}-q_{ii}^2\right)E_{i,1}^+E_i\left(E_{i,1}^+\right)^2\\
&+q_{pi}^3\left(q_{ii}^6-q_{ii}^3+q_{ii}+q_{ii}^2+2q_{ii}^3-q_{ii}-q_{ii}^2\right)E_i\left(E_{i,1}^+\right)^3\\
=&-\left(1+q_{ii}^{-3}\right)\left(\left(E_{i,1}^+\right)^3E_i-q_{pi}\left(q_{ii}+q_{ii}^2+q_{ii}^3\right)\left(E_{i,1}^+\right)^2E_iE_{i,1}^+\right.\\
&\left.+q_{pi}^2\left(q_{ii}^3+q_{ii}^4+q_{ii}^5\right)E_{i,1}^+E_i\left(E_{i,1}^+\right)^2-q_{pi}^3q_{ii}^6E_i\left(E_{i,1}^+\right)^3\right).
\end{aligned}
$$

Again, the last expression coincides with the one in Eq. (7.13). This finishes the proof of Eq. (7.12) and, with it, the proof of Eq. (7.10).

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